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The Significance of Negative Frequencies in Spectrum Analysis

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***Abstract*—In EMC measurements spectra generated by pulsed signals are often measured. To calculate the spectra theoretically, Fourier methods are usually used. The Fourier transform yields a spectrum which contains positive and negative frequencies. The exponential form of the Fourier series also yields a spectrum which contains positive and negative frequencies. However, the trigonometric form of the Fourier series yields a spectrum containing only positive frequencies. Since there seems to be some doubt about the physical interpretation of negative frequencies, the relationship between the double sided spectrum of the exponential Fourier series and the single sided spectrum of the trigonometric Fourier series is shown. The reasoning is then extended to the Fourier transform to show the relationship between the double and single sided spectrum obtained by the Fourier transform. A method of obtaining spectral levels for positive frequencies only using the Fourier transform is shown.**

IN EMC WORK it is often necessary to measure pulsed spectra. In theoretical studies associated with the measurements, Fourier analysis has been used for many years to predict the spectra theoretically.

In Fourier analysis one can use Fourier transforms or Fourier series. The Fourier series gives a spectrum which contains discrete spectrum lines. It is usually cumbersome to perform mathematical operations on each spectrum line to determine the spectral characteristics of a pulsed signal or to determine the output of circuits whose input consists of pulsed signals. It has been found much more convenient to use the spectrum of a single isolated pulse to determine the output waveshape or other signal parameters of a circuit whose input is a pulsed signal. The Fourier transform gives the spectral level of a single isolated pulse. If the actual input to the circuit is a repetitively pulsed signal, the output signal due to a single pulse of the pulse train can be found by the Fourier transform method provided that the circuit has no residual current or voltage from the previous pulse.

If any standard textbook on Fourier transforms were investigated, it could be seen that the spectra found by means of Fourier transforms contain negative and positive frequencies. The question arises, what are negative frequencies and what significance do they have in spectrum analysis? In RF measurements negative frequencies cannot be defined. Are the negative frequencies merely discarded and looked upon as mathematical curiosities?

The negative frequencies cannot be discarded mathematically. The following exposition will help shed some light on the significance and the definition of negative frequencies in spectrum analysis.

Let us first examine the Fourier series in exponential form. It will be shown that the spectrum described by the Fourier series has negative frequency components in the same sense as the spectrum described by the Fourier transform. The relationship between the negative frequencies in the two spectrums will be shown later.

The Fourier series in exponential form is

$$f=(t)\sum_{n=-\infty}^{\infty}\alpha_n\epsilon^{jn\omega_r t} \tag{1}$$

Manuscript received May 13, 1967; revised August 11, 1967. This work was supported under Air Force Contract 19(628)-5049.
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where

$f(t)$ = any periodic function of time
$\omega_r = 2\pi f_r$
f_r = frequency of periodic function

$$\alpha_n = \frac{E}{T}\int_{-T/2}^{T/2} f(t)\epsilon^{-jn\omega_r t}dt \tag{2}$$

E = maximum amplitude of periodic function of time
T = period of periodic function.

An examination of (1) will show that a periodic function of time consists of an infinite sum of discrete spectral lines; the frequency of each line is nf_r. Since n has all integer values from minus infinity to plus infinity, the spectrum mathematically exists in the negative frequency region as well as the positive frequency region. However, when n equals zero, the frequency is zero which indicates the spectrum has a dc component. Some of the spectral lines including the dc component may have zero amplitude and thus vanish. If the nature of each spectral line is examined it can be seen that each line is a time function whose amplitude is α_n. The time varying part of the function is $\epsilon^{jn\omega_r t}$. The time function is not a sinusoidal time function but it is a rotating vector whose amplitude is α_n and whose angular frequency is $n\omega_r$. Equation (1) shows that n takes on all integer values from minus infinity to plus infinity. In the negative region the angular frequency of the rotating vector is negative. The negative frequency can be interpreted as clockwise rotation of the vector while positive frequency represents counter-clockwise rotation of the vector. There is no other significance to negative frequencies. It should be apparent now that the exponential form of the series yields spectral components which are not sinusoids but rotating vectors of various amplitudes and frequencies. Since the amplitude of each vector is constant it is not possible to show any direct physical interpretation of the spectral lines. The double-sided spectrum containing negative frequencies is only a mathematical and not a physical entity.

Most of us are familiar with the Fourier series in trigonometric form. The trigonometric form contains no negative frequencies. Further, the trigonometric form shows spectral lines to be of sinusoidal form. Sinusoids have physical as well as mathematical meaning. Within mathematical restrictions, which will be shown, the sinusoidal form of the series is exactly equivalent to the exponential form. The transition from the exponential form to the trigonometric form will be shown shortly.

To be perfectly general mathematically, a function of time can be real or complex. While a complex function of time has no physical interpretation, it has important mathematical uses in Fourier analyses which will not be covered here. The foregoing statements on going from the double-sided spectrum to the single-sided spectrum apply only to real time functions. Therefore, the restriction on going to the one-sided or positive frequency only spectrum from the two-sided spectrum is that the time function has to be real for the transition to be valid.

If a time function $f(t)$ is real, then

$$\alpha_{-n} = \overset{*}{\alpha}_n \tag{3}$$

where

$\overset{*}{\alpha}_n$ conjugate of α_n

$$\alpha_n = \frac{E}{T}\int_{-T/2}^{T/2} f(t)\epsilon^{-jn\omega_r t}dt.$$

If α_n were to be put in trigonometric form, we would obtain

$$\alpha_n = \frac{E}{T}\int_{-T/2}^{T/2} f(t)(\cos n\omega_r t - j \sin n\omega_r t)dt. \tag{4}$$

The real part of α_n will be called R_n and the imaginary part of α_n will be called X_n; therefore

$$R_n = \frac{E}{T}\int_{-T/2}^{T/2} f(t) \cos n\omega_r t \, dt \tag{5}$$

$$X_n = -\frac{E}{T}\int_{-T/2}^{T/2} f(t) \sin n\omega_r t \, dt \tag{6}$$

$$\alpha_n = R_n + jX_n \tag{7}$$

$$\overset{*}{\alpha}_n = R_n - jX_n = \alpha_{-n} \tag{8}$$

$$R_n = R_{-n}$$

$$X_n = -X_{-n}.$$

When n equals zero a new quantity α_0 will be defined as

$$\alpha_{-0} = \frac{E}{T}\int_{-T/2}^{T/2} f(t)dt. \tag{9}$$

Equation (9) defines the amplitude of the dc component of the spectrum.

The series can now be written as

$$f(t) = \alpha_0 + \sum_{n=-\infty}^{-1} \alpha_n \epsilon^{jn\omega_r t} + \sum_{n=1}^{\infty} \alpha_n \epsilon^{jn\omega_r t}. \tag{10}$$

Substituting (3) in (10) yields

$$f(t) = \alpha_0 + \sum_{n=1}^{\infty} \overset{*}{\alpha}_n \epsilon^{-jn\omega_r t} + \sum_{n=1}^{\infty} \alpha_n \epsilon^{jn\omega_r t}. \tag{11}$$

Substituting (7) and (8) in (11) yields

$$f(t) = \alpha_0 + \sum_{n=1}^{\infty} (R_n - jX_n)\epsilon^{-jn\omega_r t} + \sum_{n=1}^{\infty} (R_n + jX_n)\epsilon^{jn\omega_r t}.$$

Putting the exponential in trigonometric form yields

$$f(t) = \alpha_0 + \sum_{n=1}^{\infty} (R_n - jX_n)(\cos n\omega_r t - j \sin n\omega_r t) + \sum_{n=1}^{\infty} (R_n + jX_n)(\cos n\omega_r t + j \sin n\omega_r t)$$

$$f(t) = \alpha_0 + \sum_{n=1}^{\infty} [R_n \cos n\omega_r t - X_n \sin n\omega_r t$$

$$- j(R_n \sin n\omega_r t + X_n \cos n\omega_r t)] + \sum_{n=1}^{\infty} [R_n \cos n\omega_r t$$

$$- X_n \sin n\omega_r t + j(R_n \sin n\omega_r t + X_n \cos n\omega_r t)]$$

$$f(t) = \alpha_0 + \sum_{n=1}^{\infty} [2R_n \cos n\omega_r t - 2X_n \sin n\omega_r t]. \quad (12)$$

Equation (12) is the familiar trigonometric form of the Fourier series. It is exactly equivalent to the exponential form providing the time function is real. Equation (12) contains no negative frequencies. What has happened to the negative frequencies? The negative frequency came about by considering the time varying trigonometric functions to be made up of two vectors rotating in opposite directions. There is no physical meaning to the negative frequencies.

The Fourier transform will now be considered. If one has a single isolated pulse, the spectrum resulting therefrom can be described by means of the Fourier transform. The spectrum has no discrete lines but is continuous. The Fourier transforms describe a spectral level in volts per Hertz at every frequency.

The direct Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\epsilon^{-j\omega t} dt. \quad (13)$$

Equation (13) gives the spectral level of the spectrum at any angular frequency ω.

If the spectrum is known the time function can be found from the inverse transform which is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\epsilon^{j\omega t} d\omega. \quad (14)$$

Mathematicians consider (13) and (14) as a transform pair; (13) gives an amplitude spectrum from a time function and (14) gives a time function from an amplitude spectrum.

However, an amplitude spectrum has no physical meaning to the engineer. Furthermore, it seems very strange that one can obtain a time function by integrating an amplitude function. In (14) the amplitude function $F(\omega)$ is multiplied by a time function $\epsilon^{j\omega t}$; therefore, the integral yields a time function. The point is that the mathematicians have separated the amplitudes of the spectral components from the time function and have gotten a transform pair. Engineers would like to consider spectral components as time functions with associated amplitudes and not just as disassociated amplitudes.

Let us reconsider the exponential series

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \epsilon^{jn\omega_r t}. \quad (1)$$

If $2\pi f_r$ is substituted for ω_r, then

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \epsilon^{jn2\pi f_r t} \quad (15)$$

where

$$\alpha_n = \frac{E}{T} \int_{-T/2}^{T/2} f(t)\epsilon^{-jn2\pi f_r t} dt. \quad (16)$$

If $f(t)$ is not periodic but a single isolated pulse, the time function $f(t)$ can be written as

$$f(t) = \int_{-\infty}^{\infty} F(2\pi f)\epsilon^{j2\pi f t} df \quad (17)$$

where

$$F(2\pi f) = \int_{-\infty}^{\infty} f(t)\epsilon^{-j2\pi f t} dt. \quad (18)$$

Equations (17) and (18) are analogous to (15) and (16). It is easier to visualize the function of time in (17) as being the integral of a continuous spectrum whose amplitudes are described by (18) and whose frequency components are described by the exponential part of (17). The variable in (17) and (18) can be changed from f to ω, thus

$$\omega = 2\pi f \quad (19)$$

$$d\omega = 2\pi\, df. \quad (20)$$

Multiplying and dividing (17) by 2π yields

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(2\pi f)\epsilon^{j2\pi f t} 2\pi df. \quad (21)$$

Substituting (19) and (20) in (21) yields

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\epsilon^{j\omega t} d\omega. \quad (14)$$

Substituting (19) in (18) yields

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\epsilon^{-j\omega t} dt. \quad (13)$$

It can now be said that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\epsilon^{j\omega t}\, d\omega \quad (14)$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\epsilon^{-j\omega t} dt \quad (13)$$

and

$$F(\omega) = R(\omega) + jX(\omega) \quad (22)$$

$$R(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t\, dt \quad (23)$$

$$X(\omega) = -\int_{-\infty}^{\infty} f(t) \sin \omega t\, dt. \quad (24)$$

The spectral components are $F(\omega)\epsilon^{j\omega t}$. These components are not discrete lines but are continuous. The spectral components are time functions and mathematically exist at negative as well as positive frequencies. However, just as in the exponential series, the time functions are rotating vectors which do not change amplitude with time. Again if only real time functions are considered, then

$$F(\omega) = \overset{*}{F}(-\omega)$$
$$R(\omega) = R(-\omega)$$
$$X(\omega) = -X(-\omega).$$

Substituting (22) in (14) yields

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[R(\omega) + jX(\omega)]\epsilon^{j\omega t}d\omega. \quad (25)$$

Putting $\epsilon^{j\omega t}$ in trigonometric form yields

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[R(\omega) + jX(\omega)](\cos\omega t + j\sin\omega t)d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{R(\omega)\cos\omega t - X(\omega)\sin\omega t + j[X(\omega)\cos\omega t + R(\omega)\sin\omega t]\}d\omega$$

since

$$X(\omega) = -X(-\omega)$$

and

$$R(\omega) = R(-\omega)$$

it follows that

$$\int_{-\infty}^{\infty}[X(\omega)\cos\omega t + R(\omega)\sin\omega t]d\omega = 0.$$

Therefore

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[R(\omega)\cos\omega t - X(\omega)\sin\omega t]d\omega. \quad (26)$$

Equation (26) is the Fourier transform for the spectrum in trigonometric form. The frequency limits of the spectrum range from minus infinity to plus infinity and therefore the spectrum is double-sided. But since $R(\omega) = R(-\omega)$ and $X(\omega) = -X(-\omega)$, then

$$f(t) = \frac{1}{2\pi}\int_{0}^{\infty}[2R(\omega)\cos\omega t - 2X(\omega)\sin\omega t]d\omega. \quad (27)$$

Equation (27) is the Fourier transform for the single-sided spectrum. It contains no negative frequencies. It can be seen that the amplitudes of the spectral components of the one-sided spectrum are exactly double those of the two-sided spectrum. It must still be remembered that the foregoing applies only to real time functions.

Negative frequencies are associated with time functions representing clockwise rotating vectors and positive frequencies are associated with time functions representing counter-clockwise rotating vectors. These are mathematical representations of the frequency components of the spectrum and do not have direct physical interpretation. The mathematical combination of the two oppositely rotating vectors produces sinewaves. The sinewave components of the spectrum have physical meaning and it is these sinewaves that constitute the actual physical spectrum. It has been shown how to combine the rotating vectors mathematically to produce a spectrum of sinewaves which contain no negative frequencies. Therefore negative frequencies are mathematical tools which have no direct physical interpretation.

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