The $3x + 1$ Problem and its Generalizations

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1. Introduction

The $3x + 1$ problem, also known as the *Collatz problem, the Syracuse problem, Kakutani's problem, Hasse's algorithm*, and *Ulam's problem*, concerns the behavior of the iterates of the function which takes odd integers n to $3n+1$ and even integers n to $n/2$. The $3x+1$ Conjecture asserts that, starting from any *positive* integer n, repeated iteration of this function eventually produces the value 1.

The $3x + 1$ Conjecture is simple to state and *apparently* intractably hard to solve. It shares these properties with other iteration problems, for example that of aliquot sequences (see Guy [36], Problem B6) and with celebrated Diophantine equations such as Fermat's last theorem. Paul Erdös commented concerning the intractability of the $3x + 1$ problem: "Mathematics is not yet ready for such problems." Despite this doleful pronouncement, study of the $3x + 1$ problem has not been without reward. It has interesting connections with the Diophantine approximation of $\log_2 3$ and the distribution (mod 1) of the sequence $\{(3/2)^k : k = 1, 2, \ldots\}$, with questions of ergodic theory on the 2-adic integers $\mathbf{Z}_2$, and with computability theory — a generalization of the $3x + 1$ problem has been shown to be a computationally unsolvable problem. In this paper I describe the history of the $3x+1$ problem and survey all the literature I am aware of about this problem and its generalizations.

[1] I was first exposed to the $3x + 1$ problem in 1967 as a high school student working at the National Bureau of Standards. Afterwards I worked on it from time to time. Out of curiosity and frustration I gradually became a historian of the problem, accumulating a collection of papers about it. This survey is a happy consequence. I obtained a Ph.D. (1974) in analytic number theory at M.I.T. under the supervision of Harold Stark. I have been on the staff of AT&T Bell Laboratories since then, and have held visiting positions at the University of Maryland (mathematics) and Rutgers University (computer science). My research interests include computational complexity theory, number theory, and cryptography.

The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Perron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions [25]. In his notebook dated July 1, 1932, he considered the function

$$g(n) = \begin{cases} \frac{2}{3}n \ , & \text{if } \ n \equiv 0 (\text{mod } 3) \ , \\ \frac{4}{3}n - \frac{1}{3} \ , & \text{if } \ n \equiv 1 (\text{mod } 3) \ , \\ \frac{4}{3}n + \frac{1}{3} \ , & \text{if } \ n \equiv 2 (\text{mod } 3) \ , \end{cases}$$

which gives rise to a permutation P of the natural numbers

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ 1 & 3 & 2 & 5 & 7 & 4 & 9 & 11 & 6 & \ldots \end{pmatrix} .$$

He posed the problem of determining the cycle structure of P, and asked in particular whether or not the cycle of this permutation containing 8 is finite or infinite , i.e., whether or not the iterates $g^{(k)}$ (8) remain bounded or are unbounded [24]. I will call the study of the iterates of $g(n)$ the *original Collatz problem.* Although Collatz never published any of his iteration problems, he circulated them at the International Congress of Mathematicians in 1950 in Cambridge, Massachusetts, and eventually the original Collatz problem appeared in print ([9], [47], [62]). His original question concerning $g^{(k)}$ (8) has never been answered; the cycle it belongs to is believed to be infinite. Whatever its exact origins, the $3x+1$ problem was certainly known to the mathematical community by the early 1950's; it was discovered in 1952 by B. Thwaites [72].

During its travels the $3x+1$ problem has been christened with a variety of names. Collatz's colleague H. Hasse was interested in the $3x+1$ problem and discussed generalizations of it with many people, leading to the name *Hasse's algorithm* [40]. The name *Syracuse problem* was proposed by Hasse during a visit to Syracuse University in the 1950's. Around 1960, S. Kakutani heard the problem, became interested in it, and circulated it to a number of people. He said "For about a month everybody at Yale worked on it, with no result. A similar phenomenon happened when I mentioned it at the University of Chicago. A joke was made that this problem was part of a conspiracy to slow down mathematical research in the U.S. [45]." In this process it acquired the name *Kakutani's problem*. S. Ulam also heard the problem and circulated the problem at Los Alamos and elsewhere, and it is called *Ulam's problem* in some circles ([13], [72]).

In the last ten years the $3x+1$ problem has forsaken its underground existence by appearing in various forms as a problem in books and journals, sometimes without attribution as an unsolved problem. Prizes have been offered for its solution: \$50 by H. S. M. Coxeter in 1970, then \$500 by Paul Erdös, and more recently £1000 by B. Thwaites [72]. Over twenty research articles have appeared on the $3x+1$ problem and related problems.

In what follows I first discuss what is known about the $3x+1$ problem itself, and then discuss generalizations of the problem. I have included or sketched proofs of Theorems B, D, E, F, M and N because these results are either new or have not appeared in as sharp a form previously; the casual reader may skip these proofs.

2. The 3x+1 problem.

The known results on the $3x+1$ problem are most elegantly expressed in terms of iterations of the function

$$T(n) = \begin{cases} \dfrac{3n+1}{2} , & \text{if } \; n \equiv 1 (\bmod \; 2) \; , \\ \dfrac{n}{2} , & \text{if } \; n \equiv 0 (\bmod + 2) \; . \end{cases} \tag{2.1}$$

One way to think of the $3x+1$ problem involves a directed graph whose vertices are the positive integers and that has directed edges from n to $T(n)$. I call this graph the *Collatz graph* of $T(n)$ in honor of L. Collatz [25]. A portion of the Collatz graph of $T(n)$ is pictured in Fig. 1. A directed graph is said to be *weakly connected* if it is connected when viewed as an undirected graph, i.e., for any two vertices there is a path of edges joining them, ignoring the directions on the edges. The $3x+1$ Conjecture can be formulated in terms of the Collatz graph as follows.

3x+1 CONJECTURE (First form). *The Collatz graph of $T(n)$ on the positive integers is weakly connected.*

We call the sequence of iterates $(n, T(n), T^{(2)}(n), T^{(3)}(n), \ldots)$ the *trajectory* of n. There are three possible behaviors for such trajectories when $n > 0$.

(i). *Convergent trajectory.* Some $T^{(k)}(n) = 1$.

(ii). *Non-trivial cyclic trajectory.* The sequence $T^{(k)}(n)$ eventually becomes periodic and $T^{(k)}(n) \neq 1$ for any $k \geq 1$.

(iii). *Divergent trajectory.* $\lim_{k\to\infty} T^{(k)}(n) = \infty$.

The $3x + 1$ Conjecture asserts that all trajectories of positive n are convergent. It is certainly true for $n > 1$ that $T^{(k)}(n) = 1$ cannot occur without some $T^{(k)}(n) < n$ occurring. Call the least positive k for which $T^{(k)}(n) < n$ the *stopping time* $\sigma(n)$ of n, and set $\sigma(n) = \infty$ if no k occurs with $T^{(k)}(n) < n$. Also call the least positive k for which $T^{(k)}(n) = 1$ the *total stopping time* $\sigma_\infty(n)$ of n, and set $\sigma_\infty(n) = \infty$ if no such k occurs. We may restate the $3x+1$ Conjecture in terms of the stopping time as follows.

3x+1 Conjecture (Second form). *Every integer $n \geq 2$ has a finite stopping time.*

The appeal of the $3x + 1$ problem lies in the irregular behavior of the successive iterates $T^{(k)}(n)$. One can measure this behavior using the stopping time, the total stopping time, and the *expansion factor* $s(n)$ defined by

$$s(n) = \frac{\sup_{k\geq 0} T^{(k)}(n)}{n} \ ,$$

if n has a bounded trajectory and $s(n) = +\infty$ if n has a divergent trajectory. For example $n = 27$ requires 70 iterations to arrive at the value 1 and

$$s(27) = \frac{\sup_{k\geq 0} T^{(k)}(27)}{27} = \frac{4616}{27} \approx 171 \ .$$

Table 1 illustrates the concepts defined so far by giving data on the iterates $T^{(k)}(n)$ for selected values of n.

TABLE 1. Behavior of iterates $T^{(k)}(n)$.

n	$\sigma(n)$	$\sigma_\infty(n)$	$s(n)$
1	∞	2	2
7	7	11	3.7
27	59	70	171.
$2^{50}-1$	143	383	6.37×10^8
2^{50}	1	50	1
$2^{50}+1$	2	223	1.50
$2^{500}-1$	1828	4331	1.11×10^{88}
$2^{500}+1$	2	2204	1.50

The $3x+1$ Conjecture has been numerically checked for a large range of values of n. It is an interesting problem to find efficient algorithms to test the conjecture on a computer. The current record for verifying the $3x+1$ Conjecture seems to be held by Nabuo Yoneda at the University of Tokyo, who has reportedly checked it for all $n < 2^{40} \approx 1.2 \times 10^{12}$ [2]. In several places the statement appears that A. S. Fraenkel has checked that all $n < 2^{50}$ have a finite total stopping time; this statement is erroneous [32].

2.1. A heuristic argument.

The following heuristic probabilistic argument supports the $3x+1$ Conjecture (see [28]). Pick an odd integer n_0 at random and iterate the function T until another odd integer n_1 occurs. Then $\frac{1}{2}$ of the time $n_1 = (3n_0+1)/2$, $\frac{1}{4}$ of the time $n_1 = (3n_0+1)/4$, $\frac{1}{8}$ of the time $n_1 = (3n_0+1)/8$, and so on. If one supposes that the function T is sufficiently "mixing" that successive odd integers in the trajectory of n behave as though they were drawn at random $(\bmod\ 2^k)$ from the set of odd integers $(\bmod\ 2^k)$ for all k, then the expected growth in size between two consecutive odd integers in such a trajectory is the multiplicative factor

$$\left(\frac{3}{2}\right)^{1/2}\left(\frac{3}{4}\right)^{1/4}\left(\frac{3}{8}\right)^{1/8}\cdots = \frac{3}{4} < 1 \ .$$

Consequently this heuristic argument suggests that on average the iterates in a trajectory tend to shrink in size, so that divergent trajectories should not exist. Furthermore it suggests that the total stopping time $\sigma_\infty(n)$ is (in some average sense) a constant multiple of $\log n$. (Click here for more.)

From the viewpoint of this heuristic argument, the central difficulty of the $3x+1$ problem lies in understanding in detail the "mixing" properties of iterates of the function $T(n) \pmod{2^k}$ for all powers of 2. The function $T(n)$ does indeed have some "mixing" properties given by Theorems B and K below; these are much weaker than what one needs to settle the $3x+1$ Conjecture.

2.2. Behavior of the stopping time function.

It is Riho Terras's ingenious observation that although the behavior of the total stopping time function seems hard to analyze, a great deal can be said about the stopping time function. He proved the following fundamental result ([67], [68]), also found independently by Everett [31].

Theorem A (Terras). *The set of integers $S_k = \{n : n \text{ has stopping time} \le k\}$ has limiting asymptotic density $F(k)$, i.e., the limit*

$$F(k) = \lim_{x \to \infty} \frac{1}{x} \#\{n : n \le x \ \text{ and } \ \sigma(n) \le k\}$$

exists. In addition, $F(k) \to 1$ as $k \to \infty$, so that almost all integers have a finite stopping time.

The ideas behind Terras's analysis seem basic to a deeper understanding of the $3x + 1$ problem, so I describe them in detail. In order to do this, I introduce some notation to describe the results of the process of iterating the function $T(n)$. Given an integer n, define a sequence of $0 - 1$ valued quantities $x_i(n)$ by

$$T^{(i)}(n) \equiv x_i(n) \pmod 2 \ , \quad 0 \le i < \infty \ , \tag{2.2}$$

where $T^{(0)}(n) = n$. The results of first k iterations of T are completely described by the *parity vector*

$$v_k(n) = (x_0(n), \ldots, x_{k-1}(n)) \ , \tag{2.3}$$

since the result of k iterations is

$$T^{(k)}(n) = \lambda_k(n) n + \rho_k(n) \ , \tag{2.4}$$

where

$$\lambda_k(n) = \frac{3^{x_0(n) + \ldots + x_{k-1}(n)}}{2^k} \tag{2.5}$$

and

$$\rho_k(n) = \sum_{i=0}^{k-1} x_i(n) \frac{3^{x_{i+1}(n) + \ldots + x_{k-1}(n)}}{2^{k-i}} \ . \tag{2.6}$$

Note that in (2.5), (2.6) both λ_k and ρ_k are completely determined by the parity vector $\mathbf{v} = \mathbf{v}_k(n)$ given by (2.3); I sometimes indicate this by writing $\lambda_k(\mathbf{v})$, $\rho_k(\mathbf{v})$ (instead of $\lambda_k(n)$, $\rho_k(n)$). The formula (2.4) shows that a necessary condition for $T^{(k)}(n) < n$ is that

$$\lambda_k(n) < 1 \ , \tag{2.7}$$

since $\rho_k(n)$ is nonnegative. Terras [67] defines the *coefficient stopping time* $\omega(n)$ to be the least value of k such that (2.7) holds, and $+\infty$ if no such value of k exists. It is immediate that

$$\omega(n) \leq \sigma(n) \ . \tag{2.8}$$

The function $\omega(n)$ plays an important role in the analysis of the behavior of the stopping time function $\sigma(n)$, see Theorem C.

The formula (2.2) expresses the parity vector $\mathbf{v} = \mathbf{v}_k(n)$ as a function of n. Terra's idea is to reverse this process and express n as a function of $\mathbf{v}$.

Theorem B The function $Q_k : \mathbf{Z} \to \mathbf{Z}/2^k\mathbf{Z}$ *defined by*

$$Q_k(n) = \sum_{i=0}^{k-1} x_i(n) 2^i$$

is periodic with period 2^k. *The induced function* $\bar{Q}_k : \mathbf{Z}/2^k\mathbf{Z} \to \mathbf{Z}/2^2\mathbf{Z}$ *is a permutation, and its order is a power of 2.*

Proof B (sketch). The theorem is established by induction on k, using the inductive hypotheses:

(1) $x_i(n)$ is periodic with period 2^{i+1} for $0 \leq i \leq k-1$. In fact

$$x_i(n + 2^i) \equiv x_i(n) + 1 \pmod{2} \tag{2.9}$$

for $0 \leq i \leq k-1$.

(2) $Q_k(n)$ is periodic with period 2^k.

(3) $\lambda_k(n)$ and $\rho_k(n)$ are periodic with period 2^k.

(4) $\bar{Q}_k$ is a permutation whose order divides 2^k. Also

$$\bar{Q}_k(n + 2^{k-1}) \equiv \bar{Q}_k(n) + 2^{k-1} \pmod{2^k} . \tag{2.10}$$

I omit the details. $\square$

The cycle structure and order of the first few permutations $\bar{Q}_k$ are given in Table 2. (One-cycles are omitted.) It is interesting to observe that the order of the permutation $\bar{Q}_k$ seems to be much smaller than the upper bound 2^k proved in Theorem B. Is there some explanation of this phenomenon?

TABLE 2. Cycle structure and order of permutation $\bar{Q}_k$.

k	$\bar{Q}_k$	order
1	identity	1
2	identity	1
3	(1,5)	2
4	(1,5)(2,10)(9,13)	2
5	(1,21)(2,10)(4,20)(5,17)(7,23)(9,29,25,13)(18,26)	4
6	(1,21)(2,42)(3,35)(4.20)(5,17,37,49)	
	(7,23)(8,40)(9,29,25,13)(10,34)	
	(18,58,50,26)(19,51)(27,59)(33,53)	
	(36,52)(29,55)(41,61,57,45)	4

Theorem B allows one to associate with each vector $\mathbf{v} = (v_0, \ldots, v_{k-1}) \in (\mathbf{Z}/2\mathbf{Z})^k$ of length k a unique congruence class $S(\mathbf{v}) \pmod{2^k}$ given by

$$S(\mathbf{v}) = \{n : \mathbf{v} = (x_0(n), \ldots, x_{k-1}(n))\} .$$

The integer

$$n_0(\mathbf{v}) \equiv (\bar{Q}_k)^{-1} \left(\sum_{i=0}^{k-1} v_i 2^i \right) \pmod{2^k}$$

with $0 \leq n_0(\mathbf{v}) < 2^k$ is the minimal element in $S(\mathbf{v})$ and $S(\mathbf{v})$ is the arithmetic progression:

$$S(\mathbf{v}) = \{n_0(\mathbf{v}) + 2^k i : 0 \leq i < \infty\} \ .$$

Now I consider the relation between a vector $\mathbf{v}$ and stopping times for integers $n \in S(\mathbf{v})$. Define a vector $\mathbf{v} = (v_0, v_1, \ldots, v_{k-1})$ of length k to be *admissible* if

(1) $(v_0 + \ldots + v_{k-1}) \ln 3 < k \ln 2$,

(2) $(v_0 + \ldots + v_i) \ln 3 > (i+1) \ln 2$, when $0 \leq i \leq k-2$.

Note that all admissible vectors $\mathbf{v}$ of length k have

$$v_0 + \ldots + v_{k-1} = [k\theta] \ , \tag{2.11}$$

where $\theta = \ln 2 / \ln 3 = (\log_2 3)^{-1} \approx .63093$ and $[x]$ denotes the largest integer $\leq x$. The following result is due to Terras.

Theorem C (Terras). (a) *The set of integers with coefficient stopping time k are exactly the set of integers in those congruence classes $n \pmod{2^k}$ for which there is an admissible vector $\mathbf{v}$ of length k with $n = n_0(\mathbf{v})$.*

(b) *Let $n = n_0(\mathbf{v})$ for some vector $\mathbf{v}$ of length k. If $\mathbf{v}$ is admissible, then all sufficiently large integers congruent to $n \pmod{2^k}$ have stopping time k. If $\mathbf{v}$ is not admissible, then only finitely many integers congruent to $n \pmod{2^k}$ have stopping time k.*

Proof C The assertions made in (a) about coefficient stopping times follow from the definition of admissibility, because that definition asserts that

(i) $\lambda_k(\mathbf{v}) < 1$,

(ii) $\lambda_i(\mathbf{v}) > 1$ for $1 \le i \le k-1$.

To prove (b), first note that if $\mathbf{v}$ is admissible of length k, then

$$T^{(i)}(n) \ge \frac{3^{v_0+\ldots+v_{i-1}}}{2^i} n \ge n \quad \text{for} \quad 1 \le i \le k-1 \ ,$$

and so all elements of $S(\mathbf{v})$ have stopping time at least k. Now define $\epsilon_k > 0$ by

$$\epsilon_k = 1 - \frac{3^{[k\theta]}}{2^k} \ , \tag{2.12}$$

where $\theta = (\log_2 3)^{-1}$, and note that (2.11) implies that

$$\epsilon_k = 1 - \lambda_k(\mathbf{v}) = 1 - \frac{3^{v_0+\ldots+v_{k-1}}}{2^k}$$

for all *admissible* $\mathbf{v}$. Now for $n \in S(\mathbf{v})$ for an admissible $\mathbf{v}$, (2.4) may be rewritten as

$$T^{(k)}(n) = n + (\rho_k(\mathbf{v}) - \epsilon_k n) \ . \tag{2.13}$$

Hence when $\mathbf{v}$ is admissible, those n in $S(\mathbf{v})$ with

$$n > \epsilon_k^{-1} \rho_k(\mathbf{v}) \tag{2.14}$$

have stopping time k, and $\omega(n) = \sigma(n) = k$ in this case.

Now suppose $\mathbf{v}$ is not admissible. There are two cases, depending on whether or not some initial segment $(v_0, \ldots, v_i)$ of $\mathbf{v}$ is admissible. No initial segment of $\mathbf{v}$ is admissible if and only if

$$(2.15) \qquad (v_0 + \ldots + v_{i-1}) \log 3 > i \log 2 \quad \text{for} \quad 1 \le i \le k-1 \ ,$$

and when (2.15) holds say that $\mathbf{v}$ is *inflating*. If $\mathbf{v}$ is inflating, $\lambda_k(\mathbf{v}) > 1$ so that $T^{(k)}(n) \ge n$ for all n in $S(\mathbf{v})$ by (2.4), so that no elements of $S(\mathbf{v})$ have stopping time k or less. In the remaining case $\mathbf{v}$ has an initial segment $\mathbf{w} = (v_0, v_1, \ldots, v_i)$ with $i < k-1$ which is admissible. Now $S(\mathbf{v}) \subseteq S(\mathbf{w})$ and all sufficiently large elements of $S(\mathbf{w})$ have stopping time $i+1 < k$ by the argument just given. $\square$

Theorem C asserts that the set of integers I_k with a given coefficient stopping time k is a set of arithmetic progressions $(\bmod\ 2^k)$, which has the immediate consequence that I_k has the asymptotic density

$$d(I_k) = \lim_{x \to \infty} \frac{1}{x} \#\{n : n \le x \text{ and } \ n \in I_k\}$$

which is given by

$$d(I_k) = \frac{1}{2^k} \#\{\mathbf{v} : \mathbf{v} \text{ is admissible and of length } k\} \ .$$

Furthermore Theorem C asserts that the set

$$S_k = \{n : n \text{ has stopping time } k \ \}$$

differs from I_k by a finite set, so that S_k also has an asymptotic density which is the same as that of I_k. Consequently, Theorem C implies the first part of Theorem A, that the set of all integers with stopping time at most k have an asymptotic density $F(k)$ given by

$$F(k) = \sum_{\substack{\mathbf{v} \text{ admissible} \\ \text{length } (\mathbf{v}) \leq k}} \text{weight } (\mathbf{v}) \ , \tag{2.16}$$

where

$$\text{weight } (\mathbf{v}) = 2^{- \text{ length } (\mathbf{v})} \ .$$

Now the formula (2.16) can be used to prove the second part of Theorem A, and in fact to prove the stronger result that $F(k)$ approaches 1 at an exponential rate as $k \to \infty$.

Theorem D For all $k \geq 1$,

$$1 - F(k) = \lim_{x \to \infty} \frac{1}{x} \# \{n : n \leq x \text{ and } \sigma(n) > k\} \leq 2^{-\eta k} \ , \tag{2.17}$$

where

$$\eta = 1 - H(\theta) \approx .05004 \ldots \ . \tag{2.18}$$

Here $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$ *is the entropy function and* $\theta = (\log_2 3)^{-1}$.

Proof D Let $C = C_1 \cup C_2$, where

$$C_1 = \{\mathbf{v} : \mathbf{v} \text{ is admissible and length } (\mathbf{v}) \leq k\}$$

and

$$C_2 = \{\mathbf{v} : \mathbf{v} \text{ is inflating and length } (\mathbf{v}) = k\} \ .$$

Then C has the property that for any binary word $\mathbf{w}$ of length k there is a unique $\mathbf{v} \in C$ with $\mathbf{v}$ a prefix of $\mathbf{w}$. Now for any $\mathbf{v}$ with length $(\mathbf{v}) \leq k$

$$\text{weight } (\mathbf{v}) = \sum \text{weight}(\mathbf{w}) \ ,$$

where the sum is over all $\mathbf{w}$ of length k for which $\mathbf{v}$ is a prefix of $\mathbf{w}$. Hence

$$\sum_{\mathbf{v}\in C} \mathrm{weight}(\mathbf{v}) = \sum_{\mathrm{length}(\mathbf{w})=k} \mathrm{weight}(\mathbf{w}) = 1 \ .$$

From (2.16) this implies that

$$\sum_{\mathbf{v}\in C_2} \mathrm{weight}(\mathbf{v}) = 2^{-k}|C_2| = 1 - F(k) \ ,$$

where $|C_2|$ denotes the number of vectors in C_2. The already proved first part of Theorem A shows that

$$1 - F(k) = \lim_{x\to\infty} \frac{1}{x} \#\{n : n \le x \text{ and } \sigma(n) > k\} \ ,$$

so that to prove (2.17) it suffices to bound $|C_2|$ from above.

Now the definition (2.15) of an inflating vector implies that

$$C_2 \subseteq \left\{ \mathbf{v} : \sum_{i=0}^{k-1} v_i > k\theta \right\} \ ,$$

so that

$$|C_2| \le \sum_{j > k\theta} \binom{k}{j} \ . \tag{2.19}$$

The right side of (2.19) is just the tail of the binomial distribution. It is easily checked using Stirling's formula that for any constant $\alpha > \frac{1}{2}$ and any $\epsilon > 0$ the bound

$$\sum_{j>k\alpha} \binom{k}{j} \le k \binom{k}{[ka]} \le 2^{(H(\alpha)+\epsilon)k}$$

holds for all sufficiently large k. With more work one can obtain the more precise estimate (Ash [8], Lemma 4.7.2) that for any $\alpha > \frac{1}{2}$

$$\sum_{j>k\alpha} \binom{k}{j} \le 2^{H(\alpha)k} ,$$

which used in (2.19) implies (2.17). $\square$

Theorem D cannot be substantially improved; it can be proved that for any $\epsilon > 0$ we have

$$|C_2| \ge 2^{(H(\theta)-\epsilon)k}$$

for all sufficiently large k depending on ϵ. Hence for any $\epsilon > 0$

$$1 - F(k) \ge 2^{-(\eta+\epsilon)k}$$

holds for all sufficiently large k depending on ϵ.

2.3. What is the relation between the coefficient stopping time and the stopping time?

Theorem C shows that generally they are equal: For any fixed k at most a finite number of those n having coefficient stopping time $\omega(n) \le k$ have $\sigma(n) \ne \omega(n)$. Terras [67] and later Garner [34] conjecture that this never occurs.

Coefficient Stopping Time Conjecture. *For all $n \ge 2$, the stopping time $\sigma(n)$ equals the coefficient stopping time $\omega(n)$.*

The Coefficient Stopping Time Conjecture has the aesthetic appeal that if it is true, then the set of positive integers with stopping time k is exactly a collection of congruence classes (mod 2^k), as described by part (i) of Theorem C. Furthermore, the truth of the Coefficient Stopping Time Conjecture implies that there are no nontrivial cycles. To see this, suppose that there were a nontrivial cycle of period k and let n_0 be its smallest element, and note that $\sigma(n_0) = \infty$. Then $T^{(i)}(n_0) > n_0$ for $1 \leq i \leq k-1$ and

$$T^{(k)}(n_0) = \lambda_k(n_0)n_0 + \rho_k(n_0) = n_0 \ . \tag{2.20}$$

Now $\rho_k(n_0) \neq 0$ since n_0 isn't a power of 2, so that (2.20) implies that $\lambda_k(n_0) < 1$. Hence $\omega(n_0) \leq k$, so that $\omega(n_0) \neq \sigma(n_0)$.

The following result shows that the Coefficient Stopping Time Conjecture is "nearly true." I will use it later to bound the number of elements not having a finite stopping time.

Theorem E There is an effectively computable constant k_0 such that if $\mathbf{v}$ *is admissible of length $k \geq k_0$, then all elements of $S(\mathbf{v})$ have stopping time k except possibly the smallest element $n_0(\mathbf{v})$ of S.*

Proof E (sketch). The results of A. Baker and N. I. Feldman on linear forms in logarithms of algebraic numbers ([10], Theorem 3.1) imply that there is an effectively computable absolute constant $c_0 > 0$ such that for all $k, l \geq 1$,

$$|k \log 2 - l \log 3| \geq k^{-c_0} \ .$$

Consequently there is an effectively computable absolute constant c_1 such that for $k, l \geq c_1$ one has

$$|2^k - 3^l| \geq \frac{1}{2} 2^k k^{-c_1} \ ,$$

and (2.12) then yields

$$\epsilon_k \geq k^{-c_1} \ .$$

Since $\mathbf{v}$ is admissible, $v_0 + \ldots + v_{k-1} \leq \theta k$, where $\theta = (\log_2 3)^{-1}$ by (2.11). Therefore

$$\begin{aligned} \rho_k(\mathbf{v}) &= \sum_{i=0}^{k-1} v_i \frac{3^{v_{i+1}} + \ldots + v_{k-1}}{2^{k-i}} \leq \left(\sum_{i=0}^{[k\theta]} \frac{3^i}{2^{i+1}} \right) + k(1-\theta) \binom{3}{2}^{\theta k} \\ &\leq k 2^{(1-\theta)k} \ . \end{aligned}$$

But all elements of $S(\mathbf{v})$ except $n_0(\mathbf{v})$ exceed 2^k and

$$2^k > k^{c_1+1} 2^{(l-\theta)k} > \epsilon_k^{-1} \rho_k(\mathbf{v})$$

for all sufficiently large k, so the theorem follows by (2.14). $\square$

2.4. How many elements don't have a finite stopping time?

The results proved so far can be used to obtain an upper bound for the number of elements not having a finite stopping time. Let

$$\pi^*(x) = |\{n : n \leq x \text{ and } \sigma(n) < \infty\}| \ .$$

The following result is the sharpest known result concerning the size of the "exceptional" set of n with $\sigma(n) = \infty$.

Theorem F There is a positive constant c_1 such that

$$|\pi^*(x) - x| \leq c_1 x^{1-\eta} \ , \tag{2.21}$$

where $\eta \approx .05004\ldots$ is the constant defined in Theorem D.

Proof F Suppose $2^{k-1} \le x \le 2^k$. Then

$$|\{n : n \le x \text{ and } \sigma(n) = \infty\}| = |\pi^*(x) - x| \le S_1 + S_2 \ ,$$

where $S_1 = \#\{n \le 2^k : \omega(n) \ge k+1\}$ and $S_2 = \#\{n \le 2^k : \omega(n) \le k$ and $\omega(n) \ne \sigma(n)\}$. Now Theorem D shows that

$$S_1 \le c_2(2^k)^{1-\eta} \le 2c_2 x^{1-\eta} \ , \tag{2.22}$$

and Theorem E shows that

$$S_2 \le \#\{\mathbf{v} : \mathbf{v} \text{ admissible and length } (\mathbf{v}) \le k\} + c_3 \ ,$$

where $c_3 = \#\{n : \omega(n) \le k_0 \text{ and } \omega(n) \ne \sigma(n)\}$ is a constant by Theorem C. Now

$$\begin{aligned} \#\{\mathbf{v} : \mathbf{v} \text{ admissible and length } (\mathbf{v}) = i\} \quad &\le \quad \#\{\mathbf{v} : v_0 + \ldots + v_{i-1} = [i\theta]\} = \binom{i}{[i\theta]} \\ &\le \quad c_4 2^{(1-n)i} \end{aligned}$$

using the binomial theorem and Stirling's formula. Hence

$$S_2 \le c_5 2^{(l-\eta)k} + c_3 \le (2c_5 + c_3)x^{1-\eta} \ .$$

Then this inequality and (2.22) imply (2.21) with $c_1 = 2c_2 + c_3 + 2c_5$. $\square$

2.5. Behavior of the total stopping time function.

Much less is known about the total stopping time function than about the stopping time function. One phenomenon immediately observable from a table of the total stopping times of small integers is the occurrence of many pairs and triples of integers having the same finite total stopping time. From Figure 1 we see that $\sigma_\infty(20) = \sigma_\infty(21) = 6$, $\sigma_\infty(12) = \sigma_\infty(13) = 7$, $\sigma_\infty(84) = \sigma_\infty(85) = 8$, $\sigma_\infty(52) = \sigma_\infty(53) = 9$, and $\sigma_\infty(340) = \sigma_\infty(341) = 10$. Indeed for larger values of n, multiple consecutive values occur with the same total stopping time. For example there are 17 consecutive values of n with $\sigma_\infty(n) = 40$ for $7083 \leq n \leq 7099$. A related phenomenon is that over short ranges of n the function $\sigma_\infty(n)$ tends to assume only a few values (C. W. Dodge [70]). As an example the values of $\sigma_\infty(n)$ for $1000 \leq n \leq 1099$ are given in Table 3. Only 19 values for $\sigma_\infty(n)$ are observed, for which a frequency count is given in Table 4. Both of these phenomena have a simple explanation; they are caused by coalescence of trajectories of different n's after a few steps. For example the trajectories of $8k + 4$ and $8k + 5$ coalesce after 3 steps, for all $k \geq 0$. More generally, the large number of coalescences of numbers n_1 and n_2 close together in size can be traced to the trivial cycle (1,2), as follows. Suppose n_1 and n_2 have $\sigma_\infty(n_1) \equiv \sigma_\infty(n_2) (\bmod\ 2)$, and let $\sigma_\infty(n_1) = r_1 \geq \sigma_\infty(n_2) = r_2$. Then the trajectories of n_1 and n_2 coalesce after at most $r_1 - 1$ iterations, since $T^{(r_1-1)}(n_1) = T^{(r_1-1)}(n_2) = 2$, since the trajectory of n_2 continues to cycle around the trivial cycle. If in addition $\lambda_{r_1-1}(n_1) = \lambda_{r_1-1}(n_2)$, which nearly always happens if n_1 and n_2 are about the same size, then the trajectories of $2^{r_1-1}k + n_1$, and $2^{r_1-1}k + n_2$ coalesce after at most $r_1 - 1$ iterations, for $k \geq 0$. In particular, $\sigma_\infty(2^{r_1-1}k + n_1) = \sigma_\infty(2^{r_1-1}k + n_2)$ then holds for $k \geq 1$. In this case the original coalescence of n_1 and n_2 has produced an infinite arithmetic progression (mod 2^{r_1-1}) of coalescences. The gradual accumulation of all these arithmetic progressions of coalescences of numbers close together in size leads to the phenomena observed in Tables 3 and

4.

Although the $3x+1$ Conjecture asserts that all integers n have a finite total stopping time, the strongest result proved so far concerning the density of the set of integers with a finite total stopping time is much weaker.

TABLE 3. Values of the total stopping time $\sigma_\infty(n)$ for $1000 \le n \le 1099$.

	1000 −1009	1010 −1019	1020 −1029	1030 −1039	1040 −1049	1050 −1059	1060 −1069	1070 −1079	1080 −1089	1090 −1099
0	72	42	34	80	23	23	80	18	31	31
1	91	42	34	26	80	61	80	107	88	31
2	72	72	42	80	80	53	80	23	31	23
3	29	72	42	99	80	53	50	18	88	23
4	45	26	10	80	23	53	23	18	31	23
5	45	26	26	80	23	107	50	18	31	50
6	45	34	26	80	80	23	23	23	88	61
7	61	99	26	80	80	53	42	23	88	88
8	72	34	80	42	23	23	18	34	15	61
9	72	42	80	42	42	23	18	34	31	23

TABLE 4. Values of $\sigma_\infty(n)$ and their frequencies for $1000 \le n \le 1099$.

$\sigma_\infty(n)$	freq.	$\sigma_\infty(n)$	freq.	$\sigma_\infty(n)$	freq.	$\sigma_\infty(n)$	freq.
10	1	29	1	50	3	88	5
15	1	31	7	53	4	91	1
18	6	34	6	61	4	99	2
23	17	42	9	72	6	107	2
26	6	45	3	80	16		

Theorem G (Crandall). *Let*

$$\pi_{\mathrm{total}}(x) = |\{n : n \le x \text{ and } \sigma_\infty(n) < \infty\}| \ .$$

Then there is a positive constant c_4 such that

$$\pi_{\mathrm{total}}(x) > x^{c_4}$$

for all sufficiently large x. (Click here for the current best result.)

Assuming that the $3x + 1$ Conjecture is true, one can consider the problem of determining the *expected size* of the total stopping time function $\sigma_\infty(n)$. Crandall [28] and Shanks [63] were guided by probabilistic heuristic arguments (like the one described earlier) to conjecture that the average order of $\sigma_\infty(n)$ should be a constant times $\ln n$; more precisely, that

$$\frac{1}{x}\sum_{n=1}^{X}\sigma_\infty(n) \sim 2\left(\ln\frac{4}{3}\right)^{-1}\ln x \ .$$

(Click here and here for further information on $\sigma_\infty(n)$.)

A modest amount of empirical evidence supports these conjectures, see [28].

2.6. Are there non-trivial cycles?

A first observation is that there are other cycles if negative integers are allowed in the domain of the function. There is a cycle of period 1 starting from $n = -1$, and there are cycles of length 3 and 11 starting from $n = -5$ and $n = -17$, respectively. Böhm and Sontacchi [13] conjecture that these cycles together with the cycles starting with $n = 0$ and $n = 1$ make up the entire set of cycles occurring under iteration of $T(n)$ applied to the integers $\mathbf{Z}$. Several authors have proposed the following conjecture ([13], [28], [41], [67]).

Finite Cycles Conjecture. *There are only a finite number of distinct cycles for the function $T(n)$ iterated on the domain $\mathbf{Z}$.*

One can easily show that for any given length k there are only a finite number of integers n that are periodic under iteration by T with period k, in fact at most 2^k such integers, as observed by Böhm and Sontacchi [13]. To see this, substitute the equation (2.4) into

$$T^{(k)}(n) = n \ , \quad n \in \mathbf{Z} \tag{2.23}$$

to obtain the equation

$$\left(1 - \frac{3^{x_0+\ldots+x_{k-1}}}{2^k}\right) n = \frac{3^{x_0+\ldots+x_{k-1}}}{2^k} \sum_{i=0}^{k-1} x_i \frac{2^i}{3^{x_0+\ldots+x_i}} \ . \tag{2.24}$$

There are only 2^k choices for the $0-1$ vector $\mathbf{v} = (x_0, \ldots, x_{k-1})$, and for each choice of $\mathbf{v}$ the equation (2.24) determines a unique rational solution $n = n(\mathbf{v})$. Consequently there are at most 2^k solutions to (2.23). Böhm and Sontacchi also noted that this gives an (inefficient) finite procedure for deciding if there are any cycles of a given length k, as follows: Determine the rational number $n(\mathbf{v})$ for each of the 2^k vectors $\mathbf{v}$, and for each $n(\mathbf{v})$ which is an integer test if (2.23) holds.

The argument of Böhm and Sontacchi is a very general one that makes use only of the fact that the necessary condition (2.24) for a cycle has a unique solution when the values x_i are fixed. In fact, considerably more can be proved about the nonexistence of nontrivial cyclic trajectories using special features of the necessary condition (2.24). For example, several authors have independently found a much more efficient computational procedure for proving the nonexistence of nontrivial cyclic trajectories of period $\leq k$; it essentially makes use of the inequality

$$(1 - \lambda_k(\mathbf{v}))n \leq \rho_k(\mathbf{v}) \ ,$$

which must hold for $\mathbf{v} = (x_0, x_1, \ldots, x_k)$ satisfying (2.24). This approach also allows one to check the truth of the Coefficient Stopping Time Conjecture for all n with $\omega(n) \leq k$. The basic result is as follows.

Theorem H (Terras). *For each k there is a finite bound $M(k)$ given by*

$$M(k) = \max\{\epsilon_i^{-1}\rho_i(\mathbf{v}) : \mathbf{v} \textit{ admissible, length } (\mathbf{v}) = i \le k\} \tag{2.25}$$

such that $\omega(n) \le k$ implies that $\omega(n) = \sigma(n)$ whenever $n \ge M(k)$. Consequently:

(i) *If $\sigma(n) < \infty$ for all $n \le M(k)$, then there are no non-trivial cycles of length $\le k$.*

(ii) *If $\omega(n) = \sigma(n)$ for all $n \le M(k)$, then $\omega(n) \le k$ implies $\omega(n) = \sigma(n)$.*

Proof H The existence of the bound $M(k)$ follows immediately from (2.14), and (ii) follows immediately from this fact.

To prove (i), suppose a nontrivial cycle of length $\le k$ exists. We observed earlier that if n_0 is the smallest element in a purely periodic nontrivial cycle of length $\le k$, then $\omega(n_0) = i \le k$ and $\sigma(n_0) = \infty$. The first part of the theorem then implies that $n_0 \le M(k)$. This contradicts the hypothesis of (i). $\square$

Theorem H can be used to show the nonexistence of nontrivial cycles of small period by obtaining upper bounds for the $M(k)$ and checking that condition (i) holds. This approach has been taken by Crandall [28], Garner [34], Schuppar [61] and Terras [67]. In estimating $M(k)$, one can show that the quantities $\rho_i(\mathbf{v})$ are never very large, so that the size of $M(k)$ is essentially determined by how large

$$\epsilon_i^{-1} = \left(1 - \frac{3^{[i\theta]}}{2^i}\right)^{-1}$$

can get. The worst cases occur when $3^{[i\theta]}$ is a very close approximation to 2^i, i.e., when $i/[i\theta]$ is a very good rational approximation to $\phi = \log_2 3$. The best rational approximations to ϕ are given by the convergents p_k/q_k of the continued fraction expansion of $\phi = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, 1, 4, 3 \ldots]$. Crandall [28] uses general properties of continued fraction convergents to obtain the following quantitative result.

Theorem I (Crandall). *Let n_0 be the minimal element of a purely periodic trajectory of period k. Then*

$$k > \frac{3}{2} \min \left(q_j,\ \frac{2n_0}{q_j + q_{j+1}} \right) , \tag{2.26}$$

where p_i/q_j is any convergent of the continued fraction expansion of $\log_2 3$ *with* $j \geq 4$.

As an application, use Yoneda's bound [2] that $n_0 > 2^{40}$ and choose $j = 13$ in (2.26), noting that $q_{13} = 190737$ and $q_{14} = 10590737$, to conclude that *there are no nontrivial cycles with period length less than* 275,000.

Further information about the nonexistence of nontrivial cyclic trajectories can be obtained by treating the necessary condition (2.24) as an *nonexponential Diophantine equation.* Davidson [29] calls a purely periodic trajectory of period k a *circuit* if there is a value i for which

$$n_0 < T(n_0) < \cdots < T^{(i)}(n_0)$$

and

$$T^{(i)}(n_0) > T^{(i+1)}(n_0) > \cdots > T^{(k)}(n_0) = n_0 ,$$

i.e., the parity vector $\mathbf{v}_k(n_0) = (x_0(n_0), \ldots, x_{k-1}(n_0))$ has the special form

$$x_j(n_0) = \begin{cases} 1\ , & \text{when } \ 0 \le j \le [k\theta] - 1\ , \\ 0\ , & \text{when } \ [k\theta] \le j \le k-1\ , \end{cases} \tag{2.27}$$

where $\theta = (\log_2 3)^{-1}$. The cycle starting with $n_0 = 1$ is a circuit. Davidson observed that each solution to the exponential Diophantine equation

$$(2^{a+b} - 3^b)h = 2^a - 1\ , \quad a \ge 1 \tag{2.28}$$

gives rise to a circuit of length $k = a + b$ with $[k\theta] = b$ and $n_0 = 2^b h - 1$, and conversely. (The equation (2.28) is the necessary condition (2.24) specialized to the vector (2.27). R. Steiner [64] showed that $(a, b, h) = 1, 1, 1)$ is the only solution of (2.28), thus proving the following result.

Theorem J (Steiner). *The only cycle that is a circuit is the trivial cycle.*

Proof J (sketch). Steiner's method is to show first that any solution of (2.28) with $a \ge 4$ has the property that $(a+b)/b$ is a convergent in the continued fraction expansion of $\log_2 3$, since (2.28) implies that

$$0 < \left| \frac{a+b}{b} - \log_2 3 \right| \le \frac{1}{b \ln 2(2^b - 1)}\ . \tag{2.29}$$

He checks that this rational approximation $(a+b)/b$ is so good that it violates the effective estimates of A. Baker [[10], p. 45] for linear forms in logarithms of algebraic numbers if $b > 10^{199}$. Finally he checks that (2.29) fails to hold for all that $b < 10^{199}$ by computing the convergents of the continued fractions of $\log_2 3$ up to 10^{199}. $\square$

The most remarkable thing about Theorem J is the weakness of its conclusion compared to the strength of the methods used in its proof. The proof of Theorem J does have the merit that it shows that the coefficient Stopping Time Conjecture holds for the *infinite set* of admissible vectors $\mathbf{v}$ of the form (2.27).

2.7. Do divergent trajectories exist?

Several authors have observed that heuristic probabilistic arguments suggest that no divergent trajectories occur.

DIVERGENT TRAJECTORIES CONJECTURE. *The function $T : \mathbf{Z} \to \mathbf{Z}$ has no divergent trajectories, i.e., there exists no integer n_0 for which*

$$\lim_{k\to\infty} |T^{(k)}(n_0)| = \infty \ . \tag{2.30}$$

If a divergent trajectory $\{T^{(k)}(n_0) \ : \ 0 \le k < \infty\}$ exists, it cannot be equidistributed (mod 2). Indeed if one defines

$$N^*(k) = |\{j : j \le k \text{ and } \ T^{(j)}(n_0) \equiv 1 \pmod 2\}| \ ,$$

then it can be proved that the condition (2.30) implies that

$$\liminf_{k\to\infty} \frac{N^*(k)}{k} \ge (\log_2 3)^{-1} \approx .63097 \ . \tag{2.31}$$

Theorem F constrains the possible behavior of divergent trajectories. Indeed, associated to any divergent trajectory $D = \{T^{(k)}(n_0) : k \ge 1\}$ is the infinite set $U_D = \{n : n \in D$ and

$T^{(k)}(n) > n$ for all $k \geq 1\}$. Since $\sigma(n) = \infty$ for all $n \in U_D$, Theorem F implies that

$$|\{n \in U_D : n \leq x\}| \leq c_1 x^{1-\eta} , \tag{2.32}$$

where $\eta \approx .05004$. Roughly speaking, (2.32) asserts that the elements of a divergent trajectory cannot go to infinity "too slowly."

2.8. Connections of the $3x + 1$ problem to ergodic theory.

The study of the general behavior of the iterates of measure preserving functions on a measure space is called *ergodic theory.* The $3x + 1$ problem has some interesting connections to ergodic theory, because the function T extends to a measure-preserving function on the 2-adic integers $\mathbf{Z}_2$ defined with respect to the 2-adic measure. To explain this, I need some basic facts about the 2-adic integers $\mathbf{Z}_2$, cf. [14], [50]. The 2-adic integers $\mathbf{Z}_2$ consist of all series

$$\alpha = a_0 + a_1 2 + a_2 2^2 + \cdots \ , \quad \text{all} \quad a_i = 0 \quad \text{or} \quad 1 \quad ,$$

where the $\{a_i : 0 \leq i < \infty\}$ are called the 2-*adic digits* of α. One can define congruences $(\bmod\ 2^k)$ on $\mathbf{Z}_2$ by $\alpha \equiv \beta \pmod{2^k}$ if the first k 2-adic digits of α and β agree. Addition and multiplication on $\mathbf{Z}_2$ are given by

$$X \quad = \quad \alpha + \beta \Leftrightarrow X \ (\bmod\ 2^k) \equiv \alpha \ (\bmod\ 2^k) + \beta (\bmod\ 2^k) \ \text{ for all } k \quad ,$$

$$X \quad = \quad \alpha\beta \Leftrightarrow X \ (\bmod\ 2^k) \equiv \alpha \ (\bmod\ 2^k) \cdot \beta (\bmod\ 2^k) \ \text{ for all } \ k \ .$$

The 2-*adic valuation* $||_2$ on $\mathbf{Z}_2$ is given by $|0|_2 = 0$ and for $\alpha \neq 0$ by $|\alpha|_2 = 2^{-k}$, where a_k is the first nonzero 2-adic digit of α. The valuation $||_2$ induces a metric d on $\mathbf{Z}_2$ defined by

$$d(\alpha, \beta) = |\alpha - \beta|_2 \ .$$

As a topological space $\mathbf{Z}_2$ is compact and complete with respect to the metric d; a basis of open sets for this topology is given by the 2-*adic discs of radius* 2^{-k} about α:

$$B_k(\alpha) = \{\beta \in \mathbf{Z}_2 : \alpha \equiv \beta \pmod{2^k}\} \ .$$

Finally one may consistently define the 2-*adic measure* μ_2 on $\mathbf{Z}_2$ so that

$$\mu_2(B_k(\alpha)) = 2^{-k} \ ;$$

in particular $\mu_2(\mathbf{Z}_2) = 1$. The integers $\mathbf{Z}$ are a subset of $\mathbf{Z}_2$; for example

$$-1 = 1 + 1 \cdot 2 + 1 \cdot 2^2 + \cdots \ .$$

Now one can extend the definition of the function $T : \mathbf{Z} \to \mathbf{Z}$ given by (2.1) to $T : \mathbf{Z}_2 \to \mathbf{Z}_2$ by

$$T(\alpha) = \begin{cases} \dfrac{\alpha}{2} \ , & \text{if} \quad \alpha \equiv 0 \pmod 2 \ , \\[2ex] \dfrac{3\alpha + 1}{2} \ , & \text{if} \quad \alpha \equiv 1 \pmod 2 \ . \end{cases}$$

Ergodic theory is concerned with the extent to which iterates of a function mix subsets of a measure space. I will use the following basic concepts of ergodic theory specialized to the measure space $\mathbf{Z}_2$ with the measure μ_2. A measure-preserving function $H : \mathbf{Z}_2 \to \mathbf{Z}_2$ is *ergodic* if the only μ_2-measurable sets E for which $H^{-1}(E) = E$ are $\mathbf{Z}_2$ and the empty set, i.e., such a function does such a good job of mixing points in the space that it has no nontrivial μ_2-invariant sets. It can be shown [[39], p. 36] that an equivalent condition for ergodicity is that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \mu_2(H^{-j}(B_k(\alpha)) \cap B_l(\beta)) = \mu_2(B_k(\alpha))\mu_2(B_l(\beta)) = 2^{-(k+l)} ,$$

for all $\alpha, \beta \in \mathbf{Z}_2$ and all integers $k, l \geq 0$. This condition in turn is equivalent to the assertion that for almost all $\alpha \in \mathbf{Z}_2$ the sequence of iterates

$$\{H^i(\alpha) : i = 0, 1, 2, \ldots\}$$

is uniformly distributed $(\bmod\ 2^k)$ for all $k \geq 1$. A function $H : \mathbf{Z}_2 \to \mathbf{Z}_2$ is *strongly mixing* if

$$\lim_{N\to\infty} \mu_2(H^{-N}(B_k(\alpha)) \cap B_l(\beta)) = 2^{-(k+l)}$$

for all $\alpha, \beta \in \mathbf{Z}_2$ and all $k, l \geq 0$. Strongly mixing functions are ergodic.

The following result is a special case of a result of K. P. Matthews and A. M. Watts [51].

Theorem K The map T is a measure-preserving transformation of $\mathbf{Z}_2$ which is strongly mixing. Consequently it is ergodic, and hence for almost all $\alpha \in \mathbf{Z}_2$ the sequence

$$\{T^{(i)}(\alpha) : i = 0, 1, 2, \ldots\}$$

is uniformly distributed $(\bmod\ 2^k)$ *for all $k \geq 1$.*

Theorem K implies nothing about the behavior of T on the set of integers $\mathbf{Z}$ because it is a measure 0 subset of $\mathbf{Z}_2$. In fact, the trajectory $\{T^{(i)}(n) : i = 0, 1, 2, \ldots\}$ of any integer n can *never* have the property of the conclusion of Theorem K, for if the trajectory is eventually periodic with period k, it cannot be uniformly distributed $(\bmod\ 2^{k-1})$, while if it is a divergent trajectory, it cannot even be equidistributed (mod 2) by (2.31). Consequently, this connection of the $3x+1$ problem to ergodic theory does not seem to yield any deep insight into the $3x+1$ problem itself.

There is, however, another connection of the $3x+1$ problem to ergodic theory of $\mathbf{Z}_2$ that may conceivably yield more information on the $3x+1$ problem. For each $\alpha \in \mathbf{Z}_2$ define the 0-1 variables x_i by

$$T^{(i)}(\alpha) \equiv x_i \pmod{2} \ .$$

Now define the function $Q_\infty : \mathbf{Z}_2 \to \mathbf{Z}_2$ by $Q_\infty(\alpha) = \beta$, where

$$\beta = x_0 + x_1 2 + x_2 2^2 + \cdots \ . \tag{2.33}$$

The value $Q_\infty(\alpha)$ thus encodes the behavior of *all* the iterates of α under T.

The following result has been observed by several people, including R. Terras and C. Pomerance, but has not been explicitly stated before.

Theorem L The map $Q_\infty : \mathbf{Z}_2 \to \mathbf{Z}_2$ is a continuous, one-one, onto, and measure-preserving map on the 2-adic integers $\mathbf{Z}_2$.

Proof L This is essentially a consequence of Theorem B. Use the fact that $Q_\infty(\alpha) \equiv \bar{Q}_n(\alpha) (\bmod\ 2^n)$. For any α_1, α_2 in $\mathbf{Z}_2$, if $|\alpha_1 - \alpha_2| \le 2^{-n}$, then $\alpha_1 \equiv \alpha_2 \pmod{2^n}$, so

$$Q_\infty(\alpha_1) \equiv \bar{Q}_n(\alpha_1) \equiv \bar{Q}_n(\alpha_2) \equiv Q_\infty(\alpha_2) (\bmod\ 2^n) \ ,$$

so that $|Q_\infty(\alpha_1) - Q_\infty(\alpha_2)| \le 2^{-n}$ and Q_∞ is continuous. If $\alpha_1 \ne \alpha_2$, then $\alpha_1 \not\equiv \alpha_2 \pmod{2^n}$ for some n, so that

$$Q_\infty(\alpha_1) \equiv \bar{Q}_n(\alpha_1) \not\equiv \bar{Q}_n(\alpha_2) \equiv Q_\infty(\alpha_2) \pmod{2^n}$$

and Q_∞ is one-to-one. To see that Q_∞ is onto, given α one can find β_n so that

$$\bar{Q}_n(\beta_n) \equiv \alpha \pmod{2^n} ,$$

since $\bar{Q}_n$ is a permutation. Then $|Q_\infty(\beta_n) - \alpha|_2 \le 2^{-n}$. Now $\{\beta_n\}$ forms a Cauchy sequence in the 2-adic metric and $\mathbf{Z}_2$ is compact, hence the limiting value β of $\{\beta_n\}$ satisfies $Q_\infty(\beta) = \alpha$. Now Q_∞^{-1} is defined, and $Q_\infty(\alpha) \equiv \bar{Q}_n^{-1}(\alpha) \pmod{2^n}$ implies that Q_∞^{-1} is continuous. $\square$

The $3x+1$ Conjecture can be reformulated in terms of the function Q_∞ as follows.

3x+1 CONJECTURE (Third form). *Let $\mathbf{N}^+$ denote the positive integers. Then $Q_\infty(\mathbf{N}^+) \subseteq \frac{1}{3}\mathbf{Z}$. In fact $Q_\infty(N^+) \subseteq \frac{1}{3}\mathbf{Z} - \mathbf{Z}$.*

For example $Q_\infty(1) = \sum_{i=0}^{\infty} 2^{2n} = -1/3,\ Q_\infty(2) = -2/3$, and $Q_\infty(3) = -20/3$.

The behavior of the function Q_∞ under iteration is itself of interest. Let $\mathbf{Q}_2$ denote the set of all rational numbers having odd denominators, so that $\mathbf{Q}_2 \subseteq \mathbf{Z}_2$. The set $\mathbf{Q}_2$ consists of exactly those 2-adic integers whose 2-adic expansion is finite or eventually periodic. The Finite Cycles Conjecture is equivalent to the assertion that there is a finite odd integer M such that

$$Q_\infty(\mathbf{Z}) \subseteq \frac{1}{M}\mathbf{Z} .$$

In fact one can take $M = \prod(2^l - 1)$, where the product runs over all integers l for which there is a cycle of minimal length l. As a hypothesis for further work I advance the following conjecture.

Periodicity Conjecture. $Q_\infty(\mathbf{Q}_2) = \mathbf{Q}_2$.

For example, one may calculate that $Q_\infty(10) = -26/3$, $Q_\infty(-26/3) = -54$, $Q_\infty(-54) = -82/7$, $Q_\infty(-82/7) = ?/15$. It can be shown that if n has a divergent trajectory, then the sequence $(x_0(n), x_1(n), x_2(n), \ldots)$ cannot be eventually periodic. As a consequence the truth of the Periodicity Conjecture implies the truth of the Divergent Trajectories Conjecture.

Theorem B has a curious consequence concerning the fixed points of iterates of Q_∞.

Theorem M Suppose the kth iterate $Q_\infty^{(k)}$ of Q_∞ has a fixed point $\alpha \in \mathbf{Z}_2$ which is not a fixed point of any $Q_\infty^{(l)}$ for $1 \le l < k$. Then k is a power of 2.

Proof M By hypothesis $Q_\infty^{(k)}(\alpha) = \alpha$ and $Q_\infty^{(l)}(\alpha) = \alpha_l \neq \alpha$, for $1 \le l < k$. All the α_l's are distinct for $0 \le l \le k$, since $Q_\infty^{(l_1)}(\alpha) = Q_\infty^{(l_2)}(\alpha)$ implies $Q_\infty^{(l_1 - l_2)}(\alpha) = \alpha$, since Q_∞ is one-one and onto. Consequently one can pick m large enough so that all the residue classes $\alpha_l \pmod{2^m}$ are distinct, for $0 \le l \le k$, where $\alpha_0 = \alpha$. Now the action of $Q_\infty (\text{mod } 2^m)$ is exactly that of the permutation $\bar{Q}_m$, hence

$$\bar{Q}_m^{(l)}(\alpha (\text{mod } 2^m)) \equiv \alpha_l \pmod{2^m}$$

for $0 \le l < k$. In particular $(\alpha_0 (\text{mod } 2^m),\ \alpha_1 \pmod{2^m}, \ldots, \alpha_{k-1} \pmod{2^m})$ makes up a single cycle of the permutation $\bar{Q}_m$, hence k is a power of 2 by Theorem B. $\square$

3. Generalizations of $3x + 1$ problem.

The $3x+1$ problem can be generalized by considering other functions $U : \mathbf{N} \to \mathbf{N}$ defined on the natural numbers $\mathbf{N}$ that are similar to the function T. The functions I consider to be similar to the function T are the *periodically linear functions*, which are those functions U for which there is a finite modulus d such that the function U when restricted to any congruence class $k \pmod{d}$ is linear. Some reasons to study generalizations of the $3x + 1$ problem are that they may uncover new phenomena, they can indicate the limits of validity of known results, and they can lead to simpler, more revealing proofs. Here I discuss three directions of generalizations of the $3x + 1$ problem. These deal with algorithmic decidability questions, with the existence of stopping times for almost all integers, and with the fractional parts of $(3/2)^k$.

3.1. Algorithmic decidability questions.

J. H. Conway [26] proved the remarkable result that a simple generalization of the $3x+1$ problem is algorithmically undecidable. He considers the class $\mathbf{F}$ of periodically piecewise linear functions $g : \mathbf{N} \to \mathbf{N}$ having the structure

$$g(n) = \frac{1}{(k,d)} n \text{ if } n = k \pmod{d} \quad , \qquad \text{for} \quad 0 \le k \le d-1 \quad , \tag{3.1}$$

specified by the nonnegative integers $(d, \alpha_0, \ldots, \alpha_{d-1})$. These are exactly the functions $g : \mathbf{N} \to \mathbf{N}$ such that $g(n)/n$ is periodic.

Theorem O (Conway). *For every partial recursive function f defined on a subset D of the natural numbers* $\mathbf{N}$ *there exists a function $g : \mathbf{N} \to \mathbf{N}$ such that*

(1) *$g(n)/n$ is periodic* $(\mathrm{mod}\ d)$ *for some d and takes rational values.*

(2) *There is some iterate $k \geq 1$ such that $g^{(k)}(2^m) = 2^j$ for some j if and only if m is in D.*

(3) *$g^{(k)}(2^m) = 2^{f(m)}$ for the minimal $k \geq 1$ such that $g^{(k)}(2^m)$ is a power of 2.*

Conway's proof actually gives in principle a procedure for explicitly constructing such a function g given a description of a Turing machine[2] that computes f. He carried out this procedure to find a function g associated to a particular partial recursive function f having the property that $f(2^{p_n}) = 2^{p_{n+1}}$, where p_n is the nth prime; this is described in Guy [37].

By choosing a particular partial recursive function whose domain is not a recursive subset of $\mathbf{N}$, e.g., a function f_0 that encodes the halting problem for Turing machines, we obtain the following corollary of Theorem O.

Theorem P (Conway). *There exists a particular, explicitly constructible function $g_0 : \mathbf{N} \to \mathbf{N}$ such that $g_0(n)/n$ is periodic* $(\mathrm{mod}\ d)$ *for a finite modulus d and takes rational values, for which there is no Turing machine that, when given n, always decides in a finite number of steps whether or not some iterate $g_0^{(k)}(n)$ with $k \geq 1$ is a power of 2.*

[2]Conway's proof used Minsky machines, which have the same computational power as Turing machines.

3.2. Existence of stopping times for almost all integers.

Several authors have investigated the range of validity of the result that $T(n)$ has a finite stopping time for almost all integers n by considering more general classes of periodicity linear functions. One such class $\mathbf{G}$ consists of all functions $U = U(m, d, R)$ which are given by

$$U(n) = \begin{cases} \dfrac{n}{d}\ , & \text{if } n \equiv 0 \pmod{d}\ , \\[2ex] \dfrac{mn - r}{d}\ , & \text{if } n \not\equiv 0 \pmod{d}, \text{ and } r \in R \text{ is such that} \\ & mn \equiv r \pmod{d}\ , \end{cases} \tag{3.2}$$

where m and d are positive integers with $(m, d) = 1$ and $R = \{r_i : r_i \equiv i \ (\bmod\ d), 1 \le i \le d-1\}$ is a fixed set of residue class representatives of the nonzero residue classes (mod d). The $3x+1$ function T is in the class $\mathbf{G}$. H. Möller [54] completely characterized the functions $U = U(m, d, R)$ in the set $\mathbf{G}$ which have a finite stopping time for almost all integers n. He showed they are exactly those functions for which

$$m < d^{d/(d-1)}\ . \tag{3.3}$$

E. Heppner [41] proved the following quantitative version of this result, thereby generalizing Theorem D.

Theorem Q (Heppner). *Let $U = U(m, d, R)$ be a function in the class* $\mathbf{G}$.

(i) *If $m < d^{d/(d-1)}$, then there exist real numbers $\delta_1, \delta_2 > 0$ such that for $N = [\log x / \log d]$ we have $\#\{n : n \le x$ and $U^{(N)}(n) > nx^{-\delta_1}\} = O(x^{1-\delta_2})$ as $x \to \infty$.*

(ii) *If $m > d^{d/(d-1)}$, then there exist real numbers $\delta_3, \delta_4 > 0$ such that for $N = [\log x / \log d]$ we have $\#\{n : n \le x$ and $U^{(N)}(n) < nx^{\delta_3}\} = O(x^{1-\delta_4})$ as $x \to \infty$.*

J.-P. Allouche [1] has further sharpened Theorem Q and Matthews and Watts [51], [52] have extended it to a larger class of functions.

It is a measure of the difficulty of problems in this area that even the following apparently weak conjecture is unsolved.

EXISTENCE CONJECTURE. *Let* U *be any function in the class* $\mathbf{G}$. *Then:*

(i) U *has at least one purely periodic trajectory if* $m < d^{d/(d-1)}$;

(ii) U *has at least one divergent trajectory if* $m > d^{d/(d-1)}$.

3.3. Fractional parts of $(3/2)^k$.

Attempts to understand the distribution (mod 1) of the sequence $\{(3/2)^k : 1 \le k < \infty\}$ have uncovered oblique connections with ergodic-theoretic aspects of a generalization of the $3x+1$ problem. It is conjectured that the sequence $(3/2)^k$ is uniformly distributed (mod 1).

(This conjecture seems intractable at present.)

One approach to this problem is to determine what kinds of $(\bmod\ 1)$ distributions can occur for sequences $\{(3/2)^k \xi : 1 \le k < \infty\}$, where ξ is a fixed real number. In this vein K. Mahler [49] considered the problem of whether or not there exist real numbers ξ, which he called *Z-numbers*, having the property that

$$0 \le \left\{ \left(\frac{3}{2}\right)^k \xi \right\} \le \frac{1}{2}\ , \quad k = 1, 2, 3, \dots\ , \tag{3.4}$$

where $\{x\} = x - [x]$ is the fractional part of x. He showed that the set of Z-numbers is countable, by showing that there is at most one Z-number in each interval $[n, n+1)$, for $n = 1, 2, 3, \ldots$. He went on to show that a necessary condition for the existence of a Z-number in the interval $[n, n+1)$ is that the trajectory $(n, W(n), W^{(2)}(n), \ldots)$ of n produced by the periodically linear function

$$W(n) = \begin{cases} \dfrac{3n}{2} \ , & \text{if } \ n \equiv 0 \pmod{2} \ , \\[2ex] \dfrac{3n+1}{2} \ , & \text{if } \ n \equiv 1 \pmod{2} \ , \end{cases} \tag{3.5}$$

satisfy

$$W^{(k)}(n) \not\equiv 3 \pmod{4} \ , \quad 1 \le k < \infty \ . \tag{3.6}$$

Mahler concluded from this that is unlikely that any Z-numbers exist. This is supported by the following heuristic argument. The function W may be interpreted as acting on the 2-adic integers by (3.5), and it has properties exactly analogous to the properties of T given by Theorem K. In particular, for almost all 2-adic integers α the sequence of iterates $(\alpha, W(\alpha), W^{(2)}(\alpha), \ldots)$ has infinitely many values k with $W^{(k)}(\alpha) \equiv 3 \pmod{4}$. Thus if a given $n \in \mathbf{Z}$ behaves like almost all 2-adic integers α, then (3.6) will not hold for n. Note that it is possible that all the trajectories $(n, W(n), W^{(2)}(n), \ldots)$ for $n \ge 1$ are uniformly distributed $\pmod{2^k}$ for *all* k, unlike the behavior of the function $T(n)$.

In passing, I note that the possible distributions (mod 1) of $\{(3/2)^k \xi;\ 1 \le k < \infty\}$ for real ξ have an intricate structure (see G. Choquet [16]–[22] and A. D. Pollington [57], [58]). In particular, Pollington [58] proves that there are uncountably many real numbers ξ such that

$$\frac{1}{25} \le \left\{ \left(\frac{3}{2}\right)^k \xi \right\} \le \frac{24}{25} \ ; \quad k = 1, 2, 3, \ldots ,$$

in contrast to the at most countable number of solutions ξ of (3.4).

4. Conclusion.

Is the $3x + 1$ problem intractably hard? The difficulty of settling the $3x + 1$ problem seems connected to the fact that it is a deterministic process that simulates "random" behavior. We face this dilemma: On the one hand, to the extent that the problem has structure, we can analyze it — yet it is precisely this structure that seems to prevent us from proving that it behaves "randomly." On the other hand, to the extent that the problem is structureless and "random," we have nothing to analyze and consequently cannot rigorously prove anything. Of course there remains the possibility that someone will find some hidden regularity in the $3x + 1$ problem that allows some of the conjectures about it to be settled. The existing general methods in number theory and ergodic theory do not seem to touch the $3x + 1$ problem; in this sense it seems intractable at present. Indeed all the conjectures made in this paper seem currently to be out of reach if they are true; I think there is more chance of disproving those that are false.

If the $3x+1$ problem is intractable, why should one bother to study it? One answer is provided by the following aphorism: "No problem is so intractable that something interesting cannot be said about it." Study of the $3x + 1$ problem has uncovered a number of interesting phenomena; I believe further study of it may be rewarded by the discovery of other new phenomena. It also serves as a benchmark to measure the progress of general mathematical theories. For example, future developments in solving exponential Diophantine equations may lead to the resolution of the Finite Cycles Conjecture.

If all the conjectures made in this paper are intractable, where would one begin to do research on this deceptively simple problem? As a guide to doing research, I ask questions. Here are a few that occur to me: For the $3x + 1$ problem, what restrictions are there on the growth in size of members of a divergent trajectory assuming that one exists? What interesting properties does the function Q_∞ have? Is there some direct characterization of Q_∞ other than the recursive definition (2.33)?

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