Gaussianity Measures for Detecting the Direction of Causal Time Series

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IJCAI 2011, Barcelona, Spain, July 20th
Outline

1. Introduction
2. Analysis of the AR(1) Process
3. Gaussianity Measures for Directionality Detection
4. Experiments
5. Conclusions
Detecting the Direction of Causal Time Series

Motivation:

Interesting problem for the evaluation of novel methods for causal inference [Peters et al., 2009].

New insights about the asymmetries between past and future in physical systems [Janzing, 2010].

Simple setting:

1. Linear models (AR and ARMA processes).
2. Causes $X_{t-1}, X_{t-2}, \ldots$ and effect $X_t$ have the same marginals.
Reversibility in Linear Time Series

The time series $\{X_t\}_{t=-\infty}^{\infty}$ is reversible when $\{X_{t_1}, \ldots, X_{t_n}\}$ and $\{X_{-t_1}, \ldots, X_{-t_n}\}$ have the same joint distribution for all $n, t_1 < \ldots < t_n$.

Let $\{\epsilon_t\}$ be i.i.d. white noise and consider the ARMA process

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t,$$

If the process is reversible then $\{\epsilon_t\}$ is Gaussian [Weiss, 1975].

Alternative definition of reversibility [Peters et al. (2009)]

Let $\epsilon_t \perp X_{t-k}$ for all $k > 0$. The ARMA process is reversible if

$$X_t = \sum_{i=1}^{p} \phi_i X_{t+i} + \sum_{j=1}^{q} \theta_j \tilde{\epsilon}_{t+j} + \tilde{\epsilon}_t,$$

where $\{\tilde{\epsilon}_t\}$ is an i.i.d. sequence with $\epsilon_t \perp X_{t+k}$ for all $k > 0$.

An ARMA process with i.i.d. noise is reversible $\Leftrightarrow \{\epsilon_t\}$ is Gaussian.
Overview of the Contribution

Main Result

Given a linear time series with non-Gaussian innovations, the residuals of a linear fit to the time-reversed series are more Gaussian than the residuals of the correctly ordered series.

Proved for AR(1) processes. We obtain empirical evidence that it also holds for ARMA processes.

New causal inference rules (based on Gaussianity measures) for detecting the correct ordering of linear time-series data.

These rules outperform state-of-the-art methods in experiments with simulated and real-world data.
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Stationary Causal AR(1) Process

Forward model:

\[ X_t = \phi X_{t-1} + \epsilon_t, \quad |\phi| < 1, \quad |\phi| \neq 0, \]

1. \( \{\epsilon_t\} \) is i.i.d. white noise.
2. \( \epsilon_t \perp X_{t-k} \quad \forall k > 0. \)
3. \( \phi = \text{Cov}(X_t, X_{t-1}) \left[ \text{Var}(X_t) \right]^{-1}. \)

Backward model:

\[ X_t = \tilde{\phi} X_{t+1} + \tilde{\epsilon}_t, \]

1. \( \tilde{\phi} = \text{Cov}(X_t, X_{t+1}) \left[ \text{Var}(X_t) \right]^{-1} = \phi. \)
2. What are the statistical properties of \( \{\tilde{\epsilon}_t\} \)?
Characteristics of \( \{\tilde{\epsilon}_t\} \)

When \( \{\epsilon_t\} \) is Gaussian:

1. \( \{\tilde{\epsilon}_t\} \) is i.d. white noise.
2. \( \tilde{\epsilon}_t \perp X_{t+k} \ \forall k > 0. \)
3. \( \tilde{\epsilon}_t \) has also a Gaussian distribution.

When \( \{\epsilon_t\} \) is not Gaussian:

1. \( \{\tilde{\epsilon}_t\} \) is i.d. white noise.
2. \( \tilde{\epsilon}_t \) is not independent of \( X_{t+k} \ \forall k > 0. \)
3. \( \tilde{\epsilon}_t \) is more Gaussian than \( \epsilon_t \) (New Result).
The cumulants of $\tilde{\epsilon}_t$ are written in terms of the cumulants of $\epsilon_t$:

$$\kappa_n(\tilde{\epsilon}_t) = c_n(\phi) \kappa_n(\epsilon_t),$$

where

$$c_n(\phi) = (-\phi)^n + (1 - \phi^2)^n(1 - \phi^n)^{-1},$$

which satisfies $|c_n(\phi)| < 1$, $\forall n > 2$.

The Gaussian is the only distribution whose cumulants of order larger than two are zero.

What values of $\phi$ produce the strongest Gaussianisation?

$$\lim_{n \to \infty} \arg\min_{\phi} \{|c_n(\phi)|\} = \pm(\sqrt{5} - 1)/2.$$
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Analysis of the AR(1) Process

Plots of $c_n(\phi)$ for $n = 1, \ldots, 14$
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Analysis of the AR(1) Process

Gaussianization Effect
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Unbiased Estimator of $\kappa_4$

Given a sample $\{Y_i\}_{i=1}^N$, the unbiased estimator of $\kappa_4$ with lowest variance is the 4-th $k$-statistic [Kendall et al., 1994]:

$$k_4 = \frac{1}{N^{[4]}} \left[ (N^3 + N^2)S_4 - 4(N^2 + N)S_3S_1 - 3(N^2 - N)S_2^2 + 12NS_2^2S_1 - 6S_4^2 \right],$$

where $N^{[4]} = N(N - 1)(N - 2)(N - 3)$ and $S_j = \sum_{i=1}^N Y_i^j$.

Causal Inference Rule $\kappa_4$

Select the direction for which the empirical residuals have the largest value of $k_4$. 
Maximum Mean Discrepancy (MMD)

The MMD between $p$ and $q$ within the unit ball $\mathcal{F}$ in $\mathcal{H}$ is

$$\sup_{f \in \mathcal{F}} \left( \mathbb{E}_{y \sim p}[f(y)] - \mathbb{E}_{z \sim q}[f(z)] \right) = ||\mu_p - \mu_q||_{\mathcal{H}},$$

where $\mu_p = \mathbb{E}_{y \sim p}[k(y, \cdot)]$ and $\mu_q = \mathbb{E}_{z \sim q}[k(z, \cdot)]$ are the mappings of $p$ and $q$ onto $\mathcal{H}$.

MMD is 0 when $p = q$, positive otherwise [Gretton et al., 2007].

Causal Inference Rule MMD

Select the direction for which the empirical distribution of the residuals has largest MMD distance to the Gaussian.
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Benchmark Method (HSIC)

Employs a test of independence based on the Hilbert-Schmidt Independence Criterion (HSIC) [Gretton et al., 2008].

The test computes the MMD between the empirical joint distribution and the product of the empirical marginal distributions.

Recall: when $\epsilon_t$ is not Gaussian, $\tilde{\epsilon}_t$ is not independent of $X_{t+k}$, $k > 0$.

Causal Inference Rule HSIC

Select the direction in which the empirical residuals are less dependent of the previous time series values [Peters et al., 2009].
Experiments with Simulated Data

**AR(1) processes.** 1000 series of length 100.

\[ \epsilon_t \sim |Z|^r \text{sgn}(Z), \text{ where } Z \sim N(0, \sigma^2). \sigma \text{ such that std}(\epsilon_t) = 1. \]

- \( r = 1 \) The noise is Gaussian.
- \( r > 1 \) The noise is leptokurtic.
- \( r < 1 \) The noise is platykurtic and bimodal.

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**ARMA(2,2) processes.** 1000 series of length 200.

\[ X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \epsilon_t, \]

Parameter values: \( \phi_1 = 0.9, \phi_2 = -0.3, \theta_1 = -0.29 \) and \( \theta_2 = 0.5 \) used by Peters et al., 2009.
Results on Simulated Data

- AR(1) $r = 0.75$
- AR(1) $r = 1.25$
- AR(1) $r = 1.5$
- ARMA(2.2) $\phi_1 = (\sqrt{5} - 1)/2$
- ARMA(2.2) $\phi_1 = 0.3$
- ARMA(2.2) $\phi_1 = 0.9$
Experiments on Real World Data

- 1180 series of length 500 with EEG measurements.
- ARMA processes of order up to (5,5) calibrated to each direction.
- Best model selected using AIC.

Experimental protocol of Peters et al., 2009. Given the two $p$-values generated by the HSIC method, only make a decision if
  - One of them is larger than $\alpha$ and the other one is not.
  - The distance between them is larger than $\delta$.
  - Usually $\alpha$ is small and $\delta$ is large.
Results on Real World Data

\[ \alpha = 0.02 \]

\[ \alpha = 0.04 \]

\[ \alpha = 0.06 \]

\[ \alpha = 0.08 \]
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**Conclusions**

- For an AR(1) process, the reversed residuals are more **Gaussian** than the original residuals (proved using **Cumulants**).

- Using **measures of Gaussianity**, we can detect the correct direction of an AR(1) time series.

- Also seems to work very well for **ARMA** processes.

- On simulated and real-world data, we **outperform** existing state-of-the-art methods based on testing for independencies.
References

Thank you for your attention!