

Part II: Sparse Gaussian Processes

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Computational Cost of Gaussian Processes

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Computational Cost of Gaussian Processes

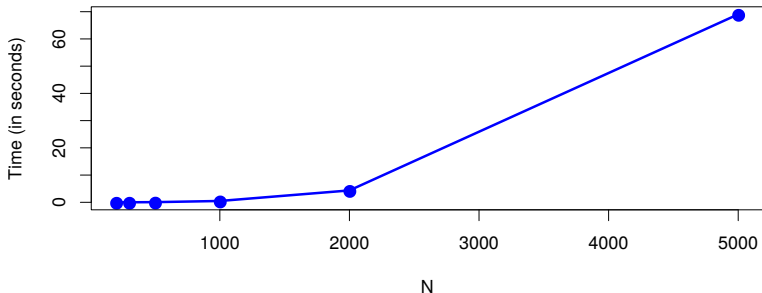
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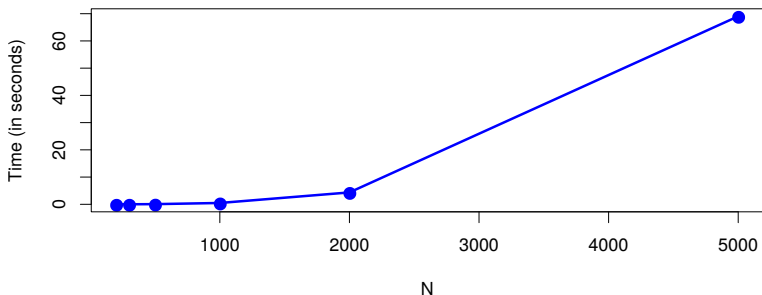
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We can handle just a few thousand data instances at most!

Run the first cells of the notebook to measure GP fitting time and complete task 1!

Improving the Cost of Gaussian Processes

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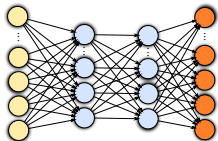
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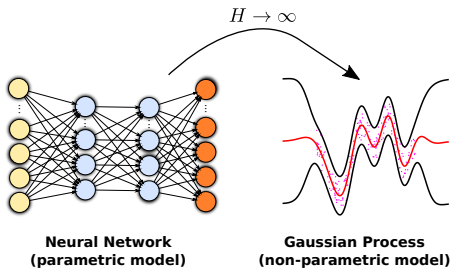
Neural Network
(parametric model)

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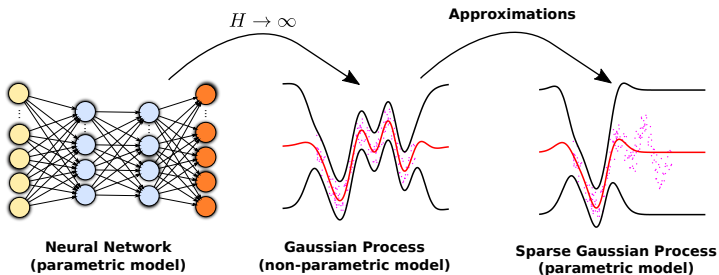


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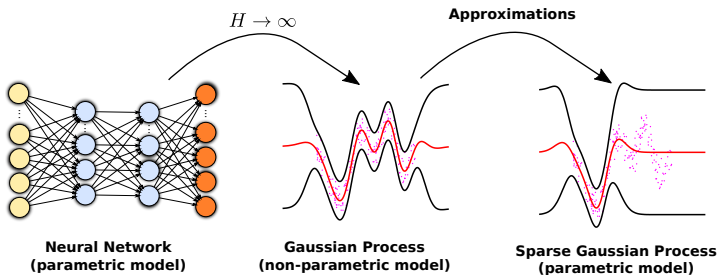


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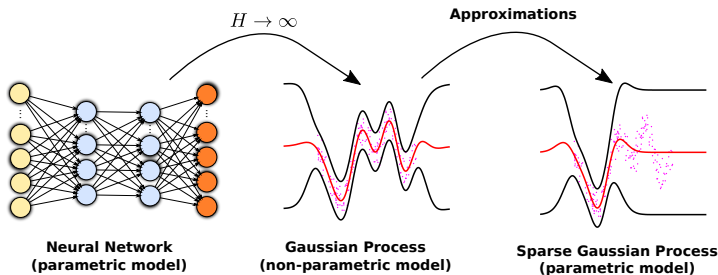
- Nyström, Random Features and FITC: approximate GP prior!

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- **Nyström, Random Features and FITC:** approximate GP prior!
- **VFE:** does approximate inference with a simplified distribution q .

The Nyström Method

Motivation: The posterior mean and covariances require $(\mathbf{I}\sigma^2 + \mathbf{\Sigma})^{-1}$.

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$$\Sigma = \begin{bmatrix} \text{[blurred matrix]} \end{bmatrix}_{N \times N} \approx \underbrace{\begin{bmatrix} \text{[blue matrix]} \end{bmatrix}}_{U}_{N \times M} \underbrace{\begin{bmatrix} \text{[diagonal matrix]} \end{bmatrix}}_{\Lambda}_{M \times M} \underbrace{\begin{bmatrix} \text{[blue matrix]} \end{bmatrix}}_{U^T}_{M \times N}$$

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The diagram illustrates the Nyström method approximation of a covariance matrix Σ . On the left, Σ is shown as an $N \times N$ matrix with a grayscale heatmap representing a dense, symmetric structure. This matrix is approximated by the product of three matrices: $\mathbf{U}$, Λ , and $\mathbf{U}^T$.
 - $\mathbf{U}$ is an $N \times M$ matrix represented by a solid blue rectangle.
 - Λ is an $M \times M$ matrix represented by a smaller rectangle containing a diagonal sequence of colored squares (pink, red, cyan, dark red, cyan, green, yellow).
 - $\mathbf{U}^T$ is an $M \times N$ matrix represented by a solid blue rectangle.
 The approximation is indicated by a tilde symbol ($\approx$) between Σ and the product of the three matrices.

The Woodbury formula gives $(\mathbf{I}\sigma^2 + \mathbf{U}\Lambda\mathbf{U}^T)^{-1}$ with cost $\mathcal{O}(M^2N)$!

Woodbury Formula

$$(\mathbf{A} + \mathbf{P}\mathbf{C}\mathbf{Q})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{P}(\mathbf{C}^{-1} + \mathbf{Q}\mathbf{A}^{-1}\mathbf{P})^{-1}\mathbf{Q}\mathbf{A}^{-1}$$

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Let us now use $\mathbf{A} = \mathbf{I}\sigma^2$, $\mathbf{P} = \mathbf{U}$, $\mathbf{Q} = \mathbf{U}^T$ and $\mathbf{C} = \mathbf{\Lambda}$.

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Note that $\mathbf{A}$ and $\mathbf{C}$ are diagonal with sizes $N \times N$ and $M \times M$!

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$\mathbf{C}^{-1} + \mathbf{Q}\mathbf{A}^{-1}\mathbf{P} = \mathbf{\Lambda}^{-1} + \mathbf{U}^\top\mathbf{U}\sigma^{-2}$ has size $M \times M$!

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$$(\mathbf{I}\sigma^2 + \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top)^{-1} = \mathbf{I}\sigma^{-2} - \sigma^{-2}\mathbf{U}(\mathbf{\Lambda}^{-1} + \mathbf{U}^\top\sigma^{-2}\mathbf{U})^{-1}\mathbf{U}^\top\sigma^{-2}$$

Computing the whole expression has cost $\mathcal{O}(NM^2)$!

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Extended input space: A function $\phi(\cdot)$ that obeys

$$\int C(\mathbf{x}, \mathbf{x}') \phi(\mathbf{x}) p(\mathbf{x}) d\mathbf{x} = \lambda \phi(\mathbf{x}'),$$

is an eigenfunction of $C(\cdot, \cdot)$ with eigenvalue λ , w.r.t., $p(\mathbf{x})$.

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Mercer's theorem:

$$C(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^{\infty} \lambda_i \phi_i(\mathbf{x}) \phi_i(\mathbf{x}').$$

An Analytic Example

Consider:

$$p(x) = \mathcal{N}(x|0, \sigma^2), \quad C(x, x') = \exp \left\{ -\frac{1}{2\ell^2} (x - x')^2 \right\} .$$

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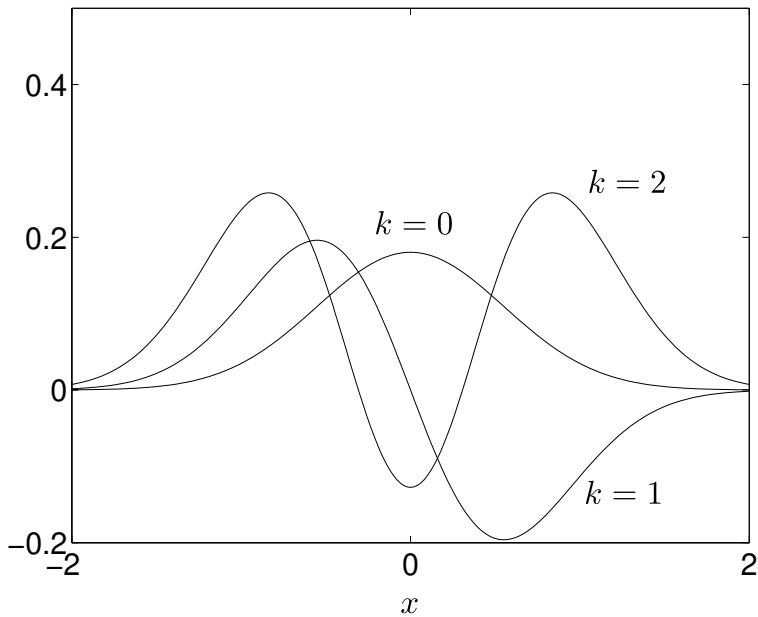
$$\lambda_k = \sqrt{\frac{2a}{A}} B^k, \quad \phi_k(x) = \exp \{ -(c - a)x^2 \} H_k(\sqrt{2c}x),$$

for $k = 0, 1, 2, \dots$, with

$$a^{-1} = 4\sigma^2, \quad b^{-1} = 2\ell^2, \quad c = \sqrt{a^2 + 2ab}, \quad A = a + b + c, \quad B = b/a,$$

and $H_k(\cdot)$, the k -th order Hermite polynomial.

Hermite Polynomials

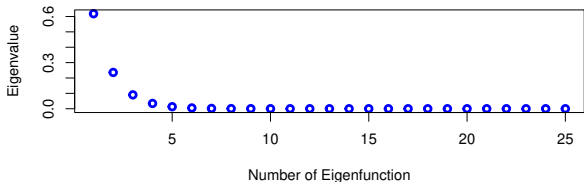


Covariance Function Approximation

Considering only the first eigenfunctions and eigenvalues is expected to give a good approximation of the covariance function!

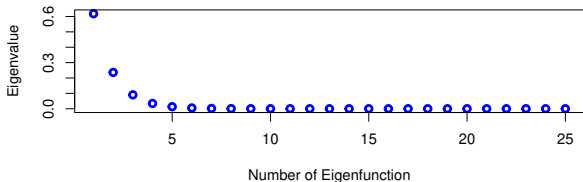
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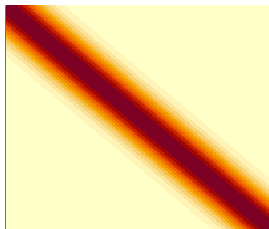


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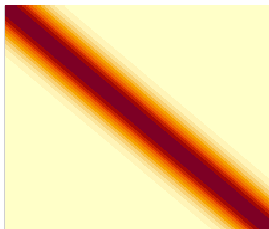
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Exact Covariance Matrix



Approx. Covariance Matrix



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$$\lambda_i \phi_i(\mathbf{x}') = \int C(\mathbf{x}, \mathbf{x}') \phi_i(\mathbf{x}) p(\mathbf{x}) d\mathbf{x} \approx \frac{1}{N} \sum_{n=1}^N C(\mathbf{x}_n, \mathbf{x}') \phi_i(\mathbf{x}_n) .$$

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For $\mathbf{x}'$ in the training set, this motivates the following eigenvalue problem:

$$\lambda_i^{\text{mat}} \mathbf{u}_i = \Sigma \mathbf{u}_i,$$

with $\Sigma_{i,j} = C(\mathbf{x}_i, \mathbf{x}_j)$. Then, we approximate $\phi_i(\mathbf{x}_j) \approx \sqrt{N}(\mathbf{u}_i)_j = \tilde{\phi}_i(\mathbf{x}_j)$, and $\lambda_i \approx \lambda_i^{\text{mat}}/N = \tilde{\lambda}_i$, which guarantees that $\Sigma = \tilde{\Phi} \tilde{\Lambda} \tilde{\Phi}^\top$.

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For an arbitrary $\mathbf{x}'$ not in the training set, then:

$$\tilde{\phi}_i(\mathbf{x}') = \frac{1}{N\lambda_i} \sum_{n=1}^N C(\mathbf{x}', \mathbf{x}_n) \phi_i(\mathbf{x}_n) \approx \frac{\sqrt{N}}{\lambda_i^{\text{mat}}} \sum_{n=1}^N C(\mathbf{x}', \mathbf{x}_n) (\mathbf{u}_i)_n = \frac{\sqrt{N}}{\lambda_i^{\text{mat}}} \Sigma(\mathbf{x}')^T \mathbf{u}_i.$$

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which results in a rank M approximation of the covariance matrix Σ :

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The inverse of $\mathbf{I}\sigma^2 + \tilde{\Sigma}$ can be efficiently computed using the Woodbury formula with cost $\mathcal{O}(NM^2)$!

Predictive Distribution

We want to compute the value of f^* at a new $\mathbf{x}^*$:

Predictive Distribution

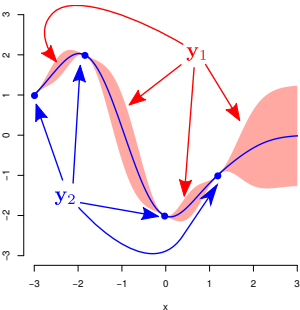
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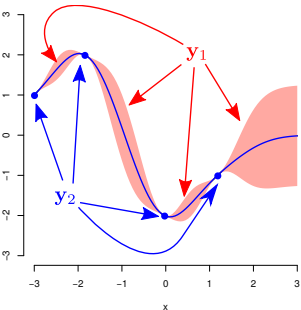
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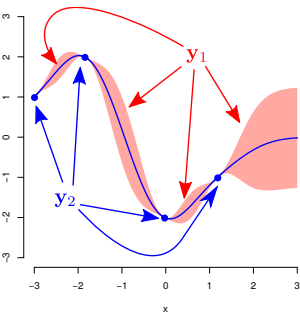


$$p(\mathbf{y}_1, \mathbf{y}_2) = \mathcal{N} \left(\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{bmatrix} \right),$$

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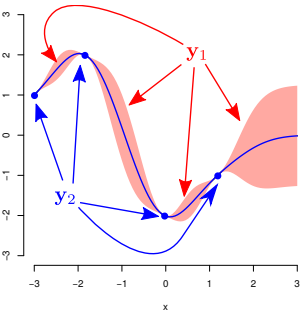


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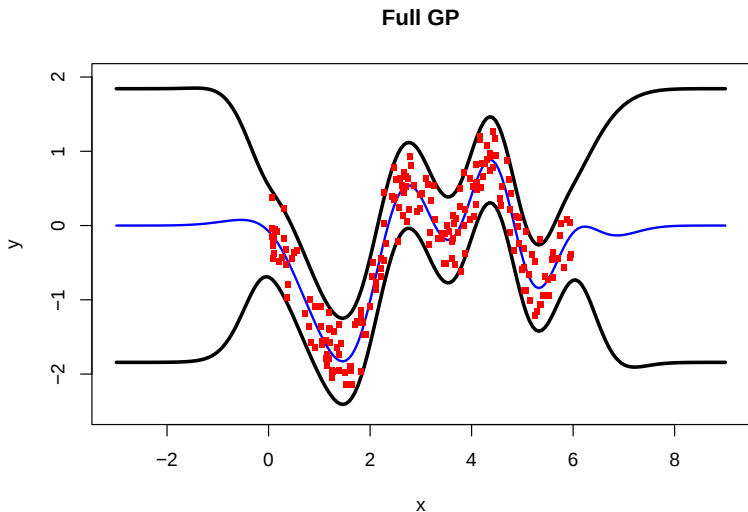


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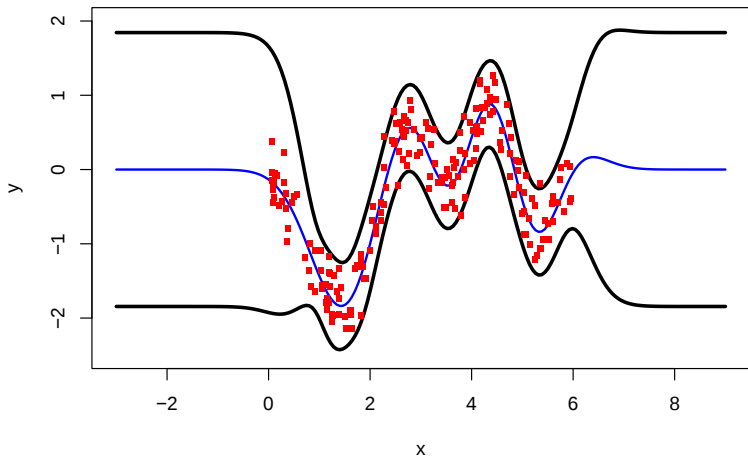
$$p(\mathbf{f}^* | \mathbf{f}) = \mathcal{N} \left(\mathbf{f}^* \middle| \Sigma_{\mathbf{f}^*\mathbf{f}} \tilde{\Sigma}_{\mathbf{f}\mathbf{f}}^{-1} \mathbf{f}, \Sigma_{\mathbf{f}^*\mathbf{f}^*} - \Sigma_{\mathbf{f}^*\mathbf{f}} \tilde{\Sigma}_{\mathbf{f}\mathbf{f}}^{-1} \Sigma_{\mathbf{f}\mathbf{f}^*}^T \right)$$

Nyström Approximation: Illustrative Example



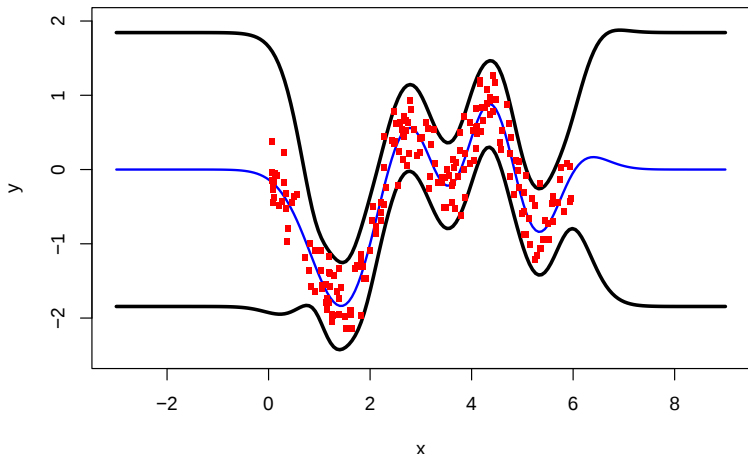
Nystrom Approximation: Illustrative Example

Nystrom GP ($M = 10$)



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The approximation is similar to the full GP in some regions!

Summary of Nyström Approximation

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- As the M points are chosen at random it may give different results.
- Since the approximation is done only over the covariance matrix of the training data, negative predictive variances are possible, but rare.

Run the cells of the notebook to fit a sparse GP using the Nyström approximation and complete task 2!

Run the cells of the notebook compare the exact and approximate predictive distribution and complete task 3!

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They can be used to approximate any stationary covariance function (only depends on the distance between points).

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$s(\mathbf{w})$ is called the spectral density of the covariance function.

Covariances as Expectations of Cosines

Due to Bochner's theorem, the covariance can be written as:

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For the squared exponential covariance function $s(\mathbf{w})$ is Gaussian!

Approximate Covariance Function

The covariance matrix can be simply approximated as:

$$\Sigma \approx \tilde{\Sigma} = \Phi\Phi^T$$

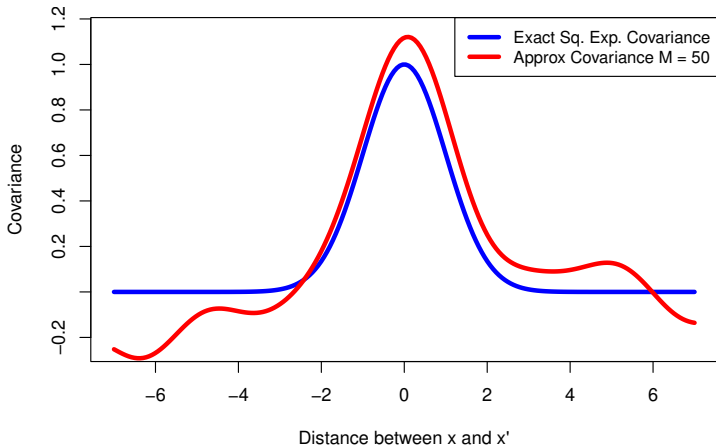
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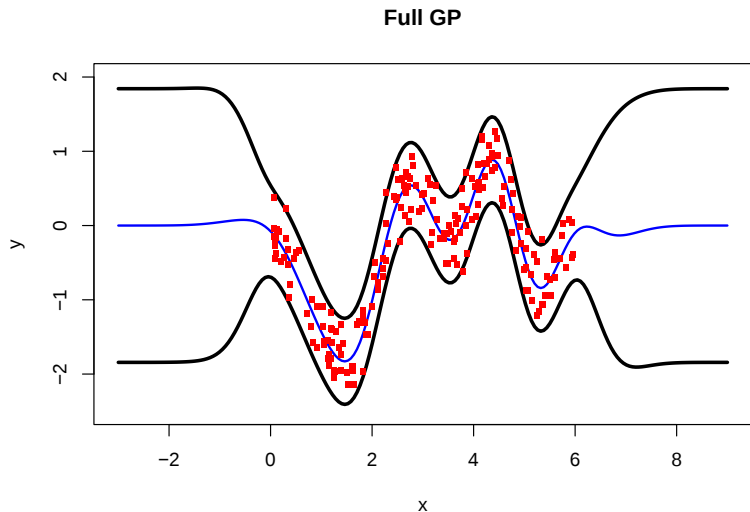
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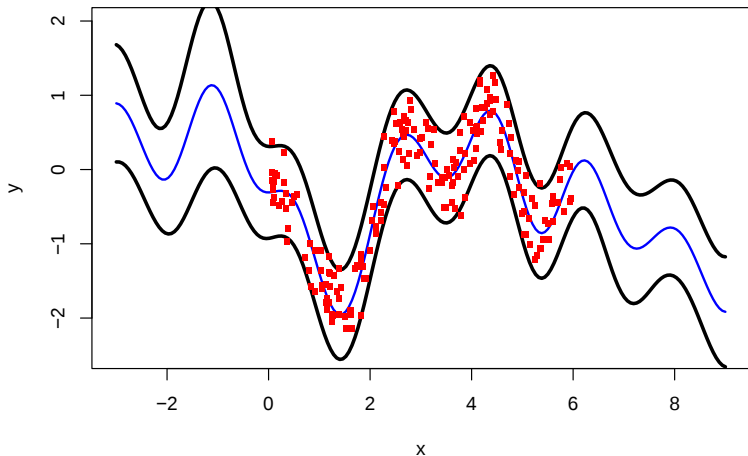
The computational cost is $\mathcal{O}(NM^2)$!

Random Features: Illustrative Example



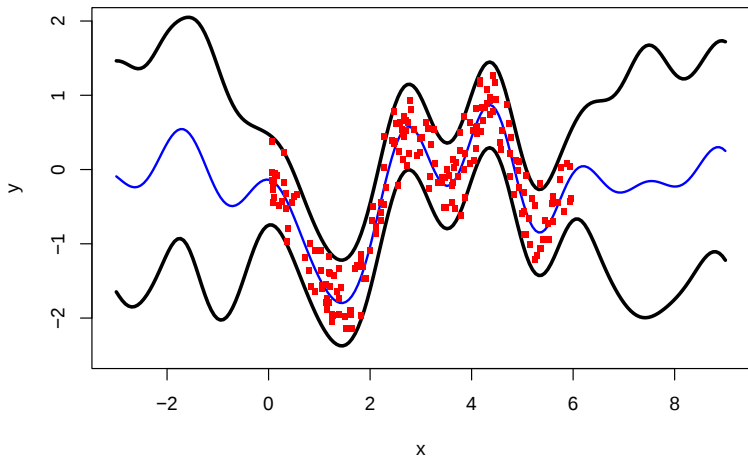
Random Features: Illustrative Example

Random Features GP ($M = 10$)



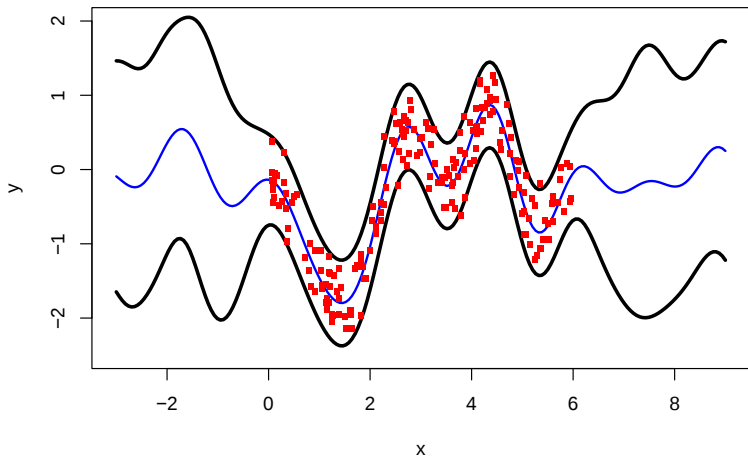
Random Features: Illustrative Example

Random Features GP ($M = 50$)



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In regions with no data the approximation may be wiggling a lot!

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- Very simple to implement!
- Equivalent to a neural network with a hidden layer with M units and cosine activations, and a Bayesian linear model in the last layer!

**Run the cells of the notebook to fit a sparse GP
using the Random Features approximation and
complete task 4!**

Full Independent Training Conditional (FITC)

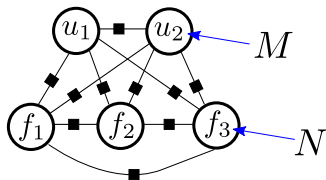
1. Extend model with $M \ll N$ inducing points and outputs at $\bar{\mathbf{X}}$.

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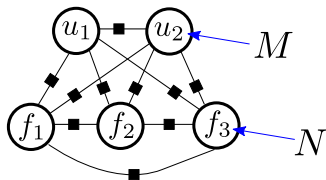
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2. Introduce conditional independences:

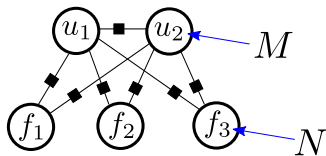
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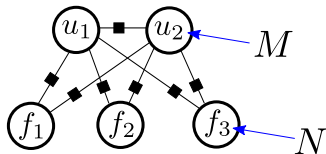
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3. Marginalize $\mathbf{u}$ to obtain an approximate GP prior for $\mathbf{f}$.

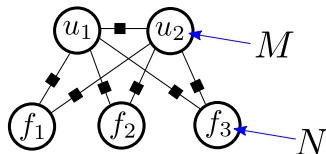
$$p(\mathbf{f}) = \int p(\mathbf{f}|\mathbf{u})p(\mathbf{u})d\mathbf{u} = \prod_{i=1}^N p(f_i|\mathbf{u})p(\mathbf{u})d\mathbf{u} = \mathcal{N}(\mathbf{f}|\mathbf{0}, \tilde{\Sigma}_{\mathbf{ff}})$$

where $\tilde{\Sigma}_{\mathbf{ff}} = \mathbf{D} + \mathbf{Q}_{\mathbf{ff}}$ with $\mathbf{D}$ diagonal and $\mathbf{Q}_{\mathbf{ff}} = \Sigma_{\mathbf{fu}}\Sigma_{\mathbf{uu}}^{-1}\Sigma_{\mathbf{uf}}$ of rank M .

Full Independent Training Conditional (FITC)

5. We make the prediction of f^* at $\mathbf{x}^*$ by considering the approximate GP prior:

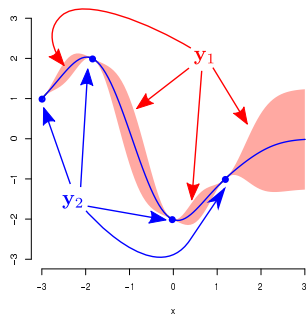
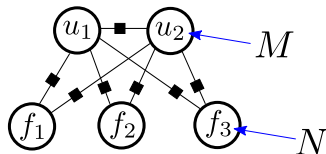
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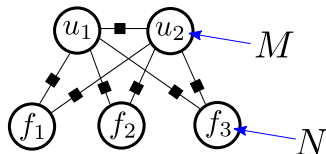
$$p(\mathbf{y}_1, \mathbf{y}_2) = \mathcal{N} \left(\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{bmatrix} \right),$$

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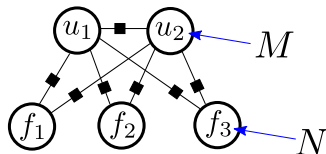


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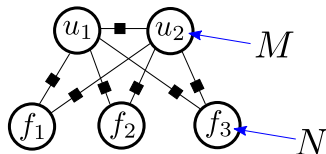
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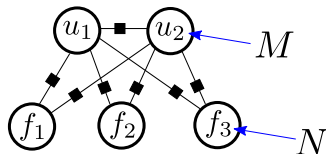
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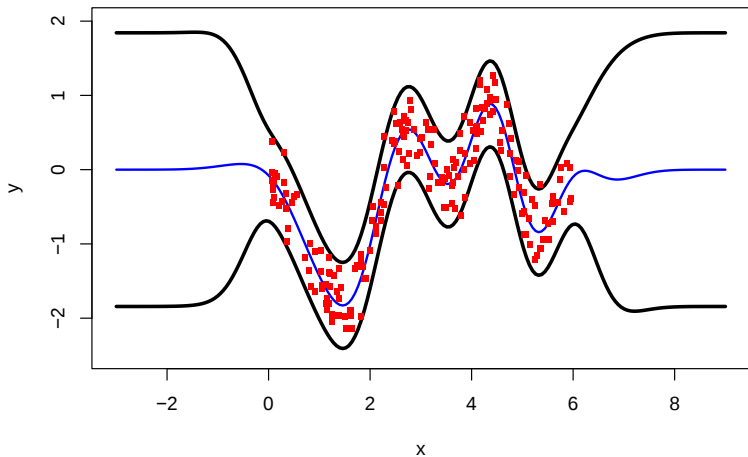
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Simply treat them as prior parameters and maximize the approximate marginal likelihood $p(\mathbf{f}|\mathbf{0}, \tilde{\Sigma}_{\mathbf{ff}})$!

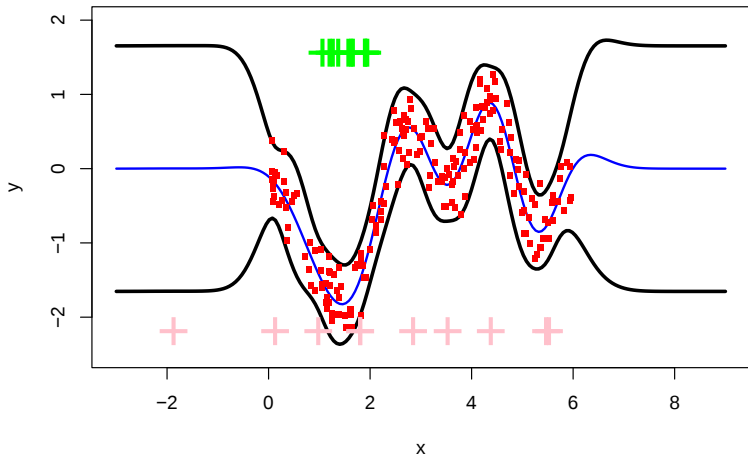
FITC: Illustrative Example

Full GP



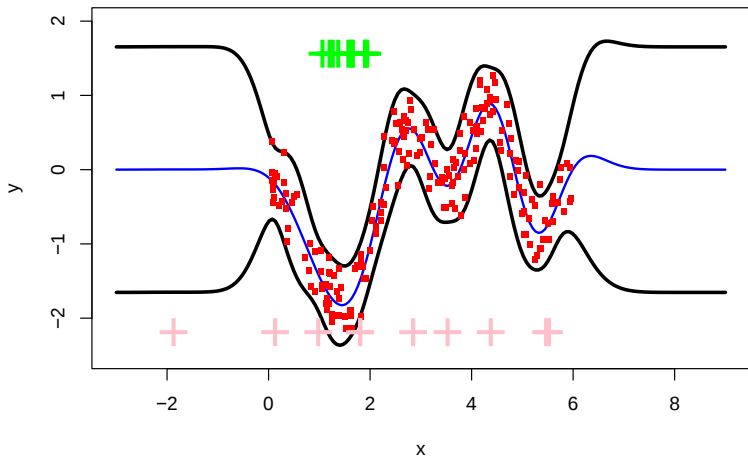
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FITC (M = 10)



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The inducing points cover the regions where the function changes!

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- The optimized inducing points spread over the input space where the latent function changes.
- Guaranteed to be exact if $M = N$ and the inducing points are not optimized and located at the training points.
- It can be understood as considering heteroscedastic (input dependent) noise!

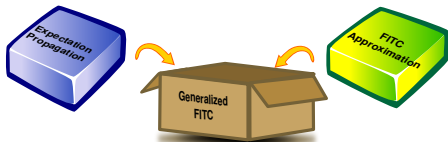
**Run the cells of the notebook to fit a sparse GP
using the FITC approximation and complete task
5!**

Generalized FITC

Combines FITC with the use of Expectation Propagation to address binary classification problems!

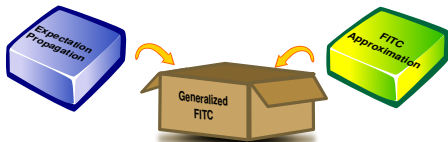
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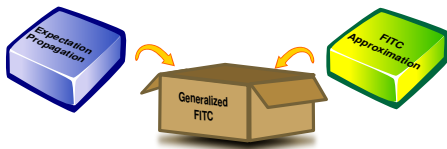


Assumes $y_i \in \{-1, 1\}$ and a probit likelihood:

$$p(y_i | f(\mathbf{x}_i)) = \phi(y_i f(\mathbf{x}_i)), \quad \phi(\cdot) \equiv \text{The c.d.f. of a standard Gaussian.}$$

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Approximates with a **Gaussian distribution** the intractable posterior:

$$p(\mathbf{f} | \mathbf{y}) = \frac{\prod_{i=1}^N \phi(y_i f(\mathbf{x}_i)) \mathcal{N}(\mathbf{f} | \mathbf{0}, \tilde{\Sigma})}{p(\mathbf{y})},$$

where $\tilde{\Sigma}$ is the **approximate FITC covariance matrix**.

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The exponential family:

$$q(\mathbf{x}) = \exp \left(\boldsymbol{\eta}^T \mathbf{u}(\mathbf{x}) - g(\boldsymbol{\eta}) \right), \quad g(\boldsymbol{\eta}) = \log \int \exp \left(\boldsymbol{\eta}^T \mathbf{u}(\mathbf{x}) \right) d\mathbf{x}$$

where $\boldsymbol{\eta}$ is a vector of natural parameters of q , $\mathbf{u}(\mathbf{x})$ are the sufficient statistics and $g(\boldsymbol{\eta})$ is a **log partition function**.

Examples of Distributions in the Exponential Family

Gaussian:

$$\mathcal{N}(x|\mu, \sigma^2) = 1/\sqrt{2\pi\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2}(x - \mu)^2 \right\}$$

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$$\boldsymbol{\eta} = (\mu/\sigma^2, 1.0/\sigma^2)^\top, \quad \mathbf{u}(x) = (x, -0.5x^2)^\top, \quad g(\boldsymbol{\eta}) = -\frac{1}{2} \log \frac{2\pi}{\eta_2} + \frac{\eta_1^2}{2\eta_2}.$$

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Most parametric distributions belong to the exponential family!

Product and Ratio of Gaussians

Consider these two Gaussian distributions:

$$p_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp \left\{ -\frac{1}{2}\sigma_1^2(x - \mu_1)^2 \right\} ,$$

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- $p_1(x)/p_2(x)$ is Gaussian with natural parameters $\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2$.
- The log-normalization constant of $p_1(x)/p_2(x)$ is $g(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) - g(\boldsymbol{\eta}_1) + g(\boldsymbol{\eta}_2)$.

KL-Divergence Minimization

Consider the KL-divergence between p and q (q in the exponential family):

$$\text{KL}(p||q) = - \int p(\mathbf{x}) \log \left\{ \frac{q(\mathbf{x})}{p(\mathbf{x})} \right\} d\mathbf{x} = g(\boldsymbol{\eta}) - \boldsymbol{\eta}^T \mathbb{E}_p[\mathbf{u}(\mathbf{x})] + \text{Const} .$$

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When **minimizing** with respect to the natural parameters $\boldsymbol{\eta}$ of q :

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If q is Gaussian, then we have to match $\mathbb{E}_q[\mathbf{x}] = \mathbb{E}_p[\mathbf{x}]$ and $\mathbb{E}_q[\mathbf{x}\mathbf{x}^T] = \mathbb{E}_p[\mathbf{x}\mathbf{x}^T]$.

Joint Approximation

EP approximates this joint distribution by a **product of simpler factors**:

$$p(\mathbf{f}, \mathbf{y}) = \prod_{i=1}^N \phi_i(y_i f(\mathbf{x}_i) | \mathcal{N}(\mathbf{f} | \mathbf{0}, \tilde{\Sigma})) = \prod_i t_i(\mathbf{f}) \approx \prod_i \tilde{t}_i(\mathbf{f}),$$

where each $\tilde{t}_i$ approximates the corresponding t_i . Each $\tilde{t}_i$ must **belong to the exponential family** but **need not be normalized**.

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The exponential family is closed under the product and $\prod_i \tilde{t}_i$ can be easily normalized to compute an approximate distribution:

$$p(\mathbf{f} | \mathbf{y}) = \frac{1}{p(\mathbf{y})} \prod_i t_i(\mathbf{f}) \approx \frac{1}{Z} \prod_i \tilde{t}_i(\mathbf{f}) = q(\mathbf{f}),$$

where $Z = \int \prod_i \tilde{t}_i(\mathbf{f}) d\mathbf{f}$ can be used to **approximate** $p(\mathbf{y})$.

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Therefore q has the same form as the approximate factors!

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Then, $\tilde{t}_j$ is updated to minimize the KL-divergence between:

$$q_{\text{new}}(\mathbf{f}) \propto \tilde{t}_j(\mathbf{f}) q^{\setminus j}(\mathbf{f}), \quad \hat{p}_j(\mathbf{f}) = \frac{1}{Z_j} t_j(\mathbf{f}) q^{\setminus j}(\mathbf{f}), \quad Z_j = \int t_j(\mathbf{f}) q^{\setminus j}(\mathbf{f}) d\mathbf{f},$$

where $q^{\setminus j}$ is fixed. This ensures that $\tilde{t}_j$ is accurate where $q^{\setminus j}$ is high.

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In practice, $\tilde{t}_j$ is found by first **minimizing** with respect to q_{new} :

$$\text{KL} \left(\frac{t_j(\mathbf{f})q^{\setminus j}(\mathbf{f})}{Z_j} \middle| q_{\text{new}}(\mathbf{f}) \right) .$$

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The refined factor $\tilde{t}_j$ is set in practice to be:

$$\tilde{t}_j(\mathbf{f}) = Z_j \frac{q_{\text{new}}(\mathbf{f})}{q^{\vee}(\mathbf{f})}, \quad \text{with} \quad \tilde{t}_j(\mathbf{f})q^{\vee}(\mathbf{f}) \propto q_{\text{new}},$$

which ensures that $\tilde{t}_j(\mathbf{f})q^{\vee}(\mathbf{f})$ and $t_j(\mathbf{f})q^{\vee}(\mathbf{f})$ **integrate the same**.

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The FITC prior results in a total cost of $\mathcal{O}(NM^2)$!

Graphical Illustration

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Graphical Illustration

Approximates $p(\mathbf{f}|\mathbf{y}) \propto t_0(\mathbf{f}) \prod_{j=1}^N t_j(\mathbf{f})$ with $q(\mathbf{f}) \propto t_0(\mathbf{f}) \prod_{j=1}^N \tilde{t}_j(\mathbf{f})$

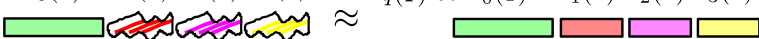
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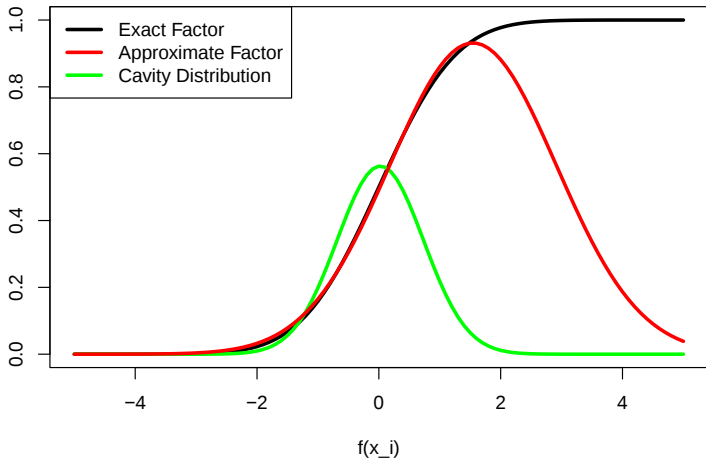
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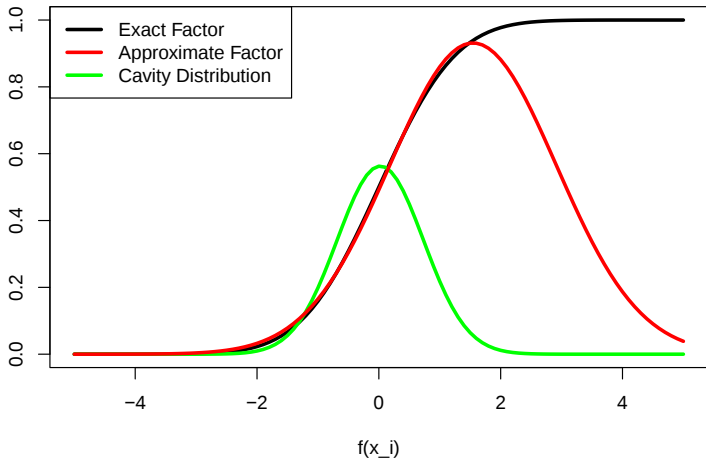
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If the exact factor already belongs to the exponential family it needs not be approximated!

GFITC: Factor Approximation



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The approximate factor is accurate in regions of high posterior probability as indicated by the cavity distribution!

Hyper-parameters and Inducing Points

They are optimized by maximizing the EP estimate of the log-marginal likelihood $\log Z \approx \log p(\mathbf{y})$.

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If EP converges the gradient of $\log Z$ w.r.t. each θ_i is zero, which allows to easily compute the gradients of $\log Z$!

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We want to compute the value of $\mathbf{f}^*$ at a new $\mathbf{x}^*$:

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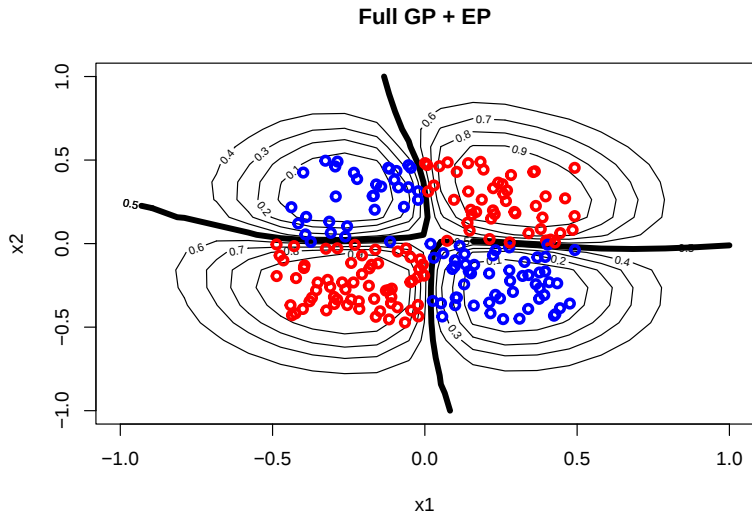
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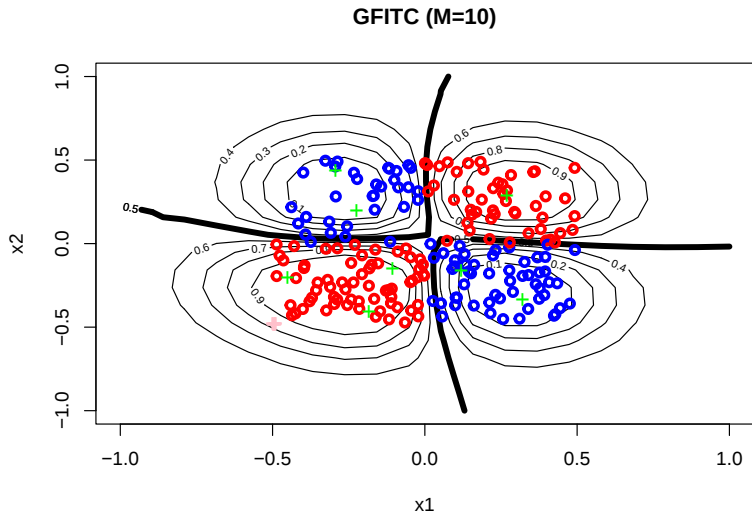
After marginalizing $\mathbf{f}$ w.r.t. $q(\mathbf{f})$, we obtain the predictive distribution:

$$\begin{aligned} p(\mathbf{f}^*|\mathbf{y}) &= \int p(\mathbf{f}^*|\mathbf{f}) q(\mathbf{f}) d\mathbf{f} \\ &= \mathcal{N}(\mathbf{f}^* | \mathbf{Q}_{\mathbf{f}^*\mathbf{f}} \tilde{\Sigma}_{\mathbf{ff}}^{-1} \tilde{\mathbf{y}}, \Sigma_{\mathbf{f}^*\mathbf{f}^*} - \mathbf{Q}_{\mathbf{f}^*\mathbf{f}} (\tilde{\Sigma}_{\mathbf{ff}} + \tilde{\Pi})^{-1} \mathbf{Q}_{\mathbf{ff}^*}) \end{aligned}$$

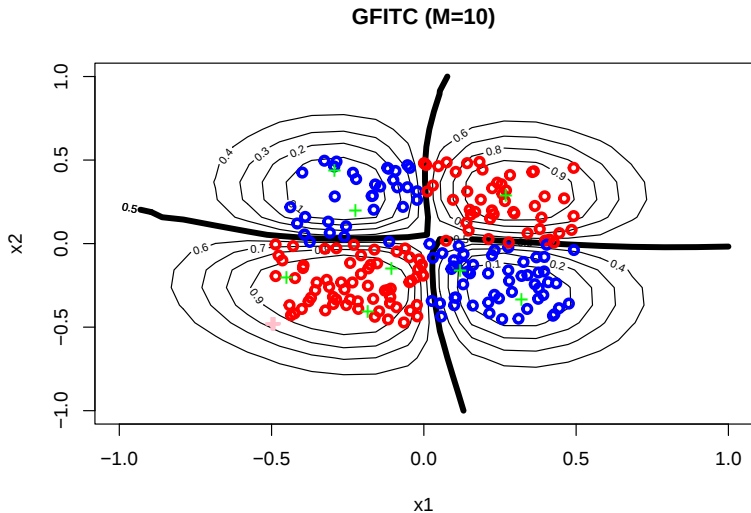
GFITC: Illustrative Example



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The inducing points spread across the input space!

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Since the GP prior is not changed it tends to perform better than the previous methods!

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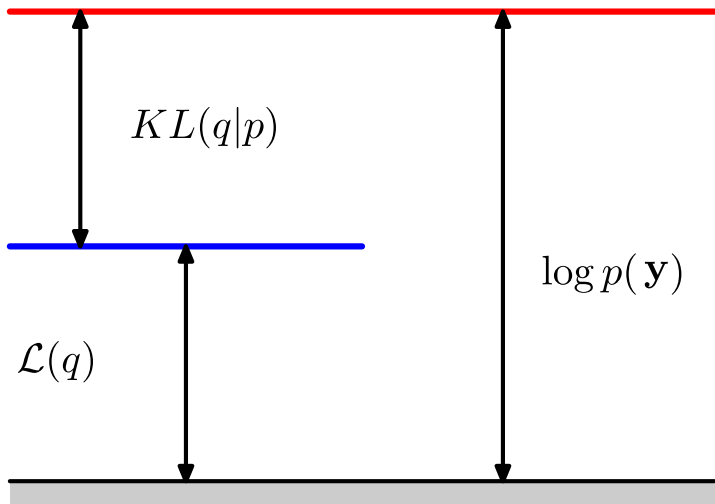
Let the target be $p(\mathbf{f}|\mathbf{y})$. Consider the decomposition of $p(\mathbf{y})$:

$$\log p(\mathbf{y}) = \mathcal{L}(q) + \text{KL}(q|p),$$

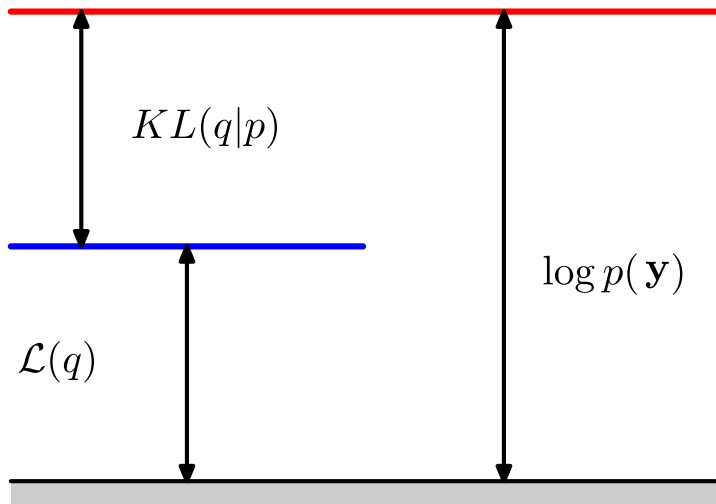
where

$$\mathcal{L}(q) = \int q(\mathbf{f}) \log \frac{p(\mathbf{f}, \mathbf{y})}{q(\mathbf{f})} d\mathbf{f}, \quad \text{KL}(q|p) = \int q(\mathbf{f}) \log \frac{q(\mathbf{f})}{p(\mathbf{f}|\mathbf{y})} d\mathbf{f}.$$

Decomposition of the Marginal Likelihood



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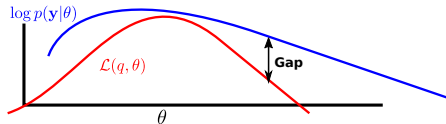


$\mathcal{L}(q)$ can be used to approximate $\log p(\mathbf{y})$ if $KL(q|p)$ is small!

Variational Free Energy (VFE)

Lower bound the log-likelihood:

$$\log p(\mathbf{y}|\theta) = \log \int p(\mathbf{y}, \mathbf{f}, \mathbf{u}|\theta) d\mathbf{f} d\mathbf{u}$$

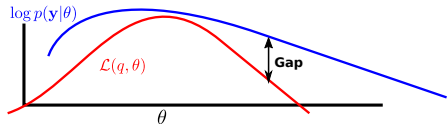


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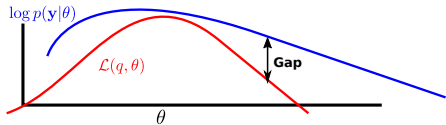


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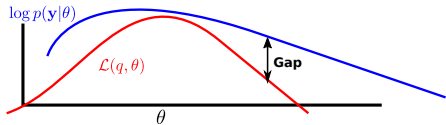
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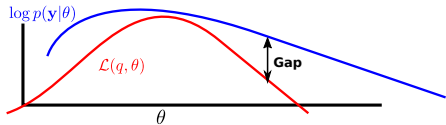
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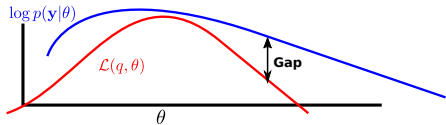
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KL $\equiv$ Kullback-Leibler divergence

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$\text{KL} \equiv$ Kullback-Leibler divergence

By maximizing $\mathcal{L}(q, \theta)$ w.r.t q we are enforcing that $q(\mathbf{f}, \mathbf{u})$ looks similar to $p(\mathbf{f}, \mathbf{u}|\mathbf{y})$ in terms of the KL!

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Consider the following approximate distribution:

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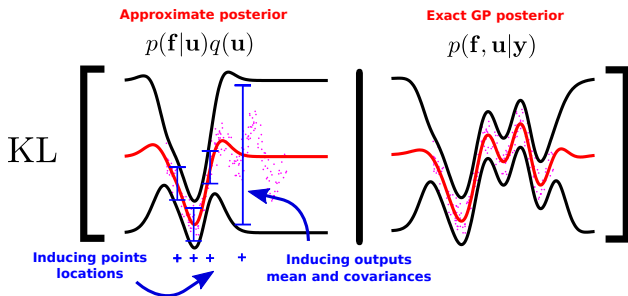


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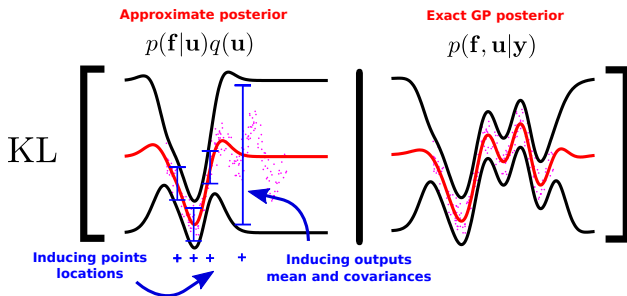


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The inducing points are now parameters of the approx. dist. q !

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- Predictions are made using $p(\mathbf{f}^* | \mathbf{u}) q(\mathbf{u})$ marginalizing out $\mathbf{u}$.

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$$p(\mathbf{f}^*, \mathbf{u}) = \mathcal{N} \left(\begin{bmatrix} \mathbf{f}^* \\ \mathbf{u} \end{bmatrix} \middle| \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \Sigma_{\mathbf{f}^* \mathbf{f}^*} & \Sigma_{\mathbf{f}^* \mathbf{u}} \\ \Sigma_{\mathbf{u} \mathbf{f}^*} & \Sigma_{\mathbf{u} \mathbf{u}} \end{bmatrix} \right)$$

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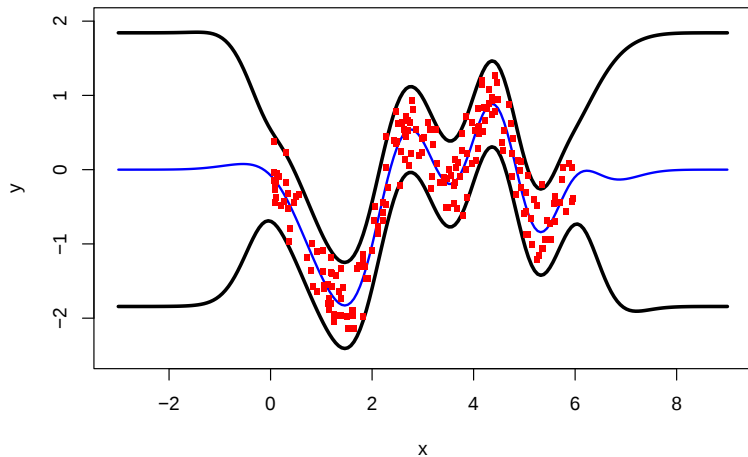
$$p(\mathbf{f}^* | \mathbf{u}) = \mathcal{N}(\mathbf{f}^* | \Sigma_{\mathbf{f}^* \mathbf{u}} \Sigma_{\mathbf{u} \mathbf{u}}^{-1} \mathbf{u}, \Sigma_{\mathbf{f}^* \mathbf{f}^*} - \Sigma_{\mathbf{f}^* \mathbf{u}} \Sigma_{\mathbf{u} \mathbf{u}}^{-1} \Sigma_{\mathbf{u} \mathbf{f}^*})$$

After marginalizing $\mathbf{u}$ w.r.t. $q(\mathbf{u})$, we obtain the predictive distribution:

$$\begin{aligned} p(\mathbf{f}^* | \mathbf{y}) &= \int p(\mathbf{f}^* | \mathbf{u}) q(\mathbf{u}) d\mathbf{u} \\ &= \mathcal{N}(\mathbf{f}^* | \Sigma_{\mathbf{f}^* \mathbf{u}} \Sigma_{\mathbf{u} \mathbf{u}}^{-1} \mathbf{m}, \Sigma_{\mathbf{f}^* \mathbf{f}^*} - \Sigma_{\mathbf{f}^* \mathbf{u}} (\Sigma_{\mathbf{u} \mathbf{u}}^{-1} - \Sigma_{\mathbf{u} \mathbf{u}}^{-1} \mathbf{S} \Sigma_{\mathbf{u} \mathbf{u}}^{-1}) \Sigma_{\mathbf{u} \mathbf{f}^*}) \end{aligned}$$

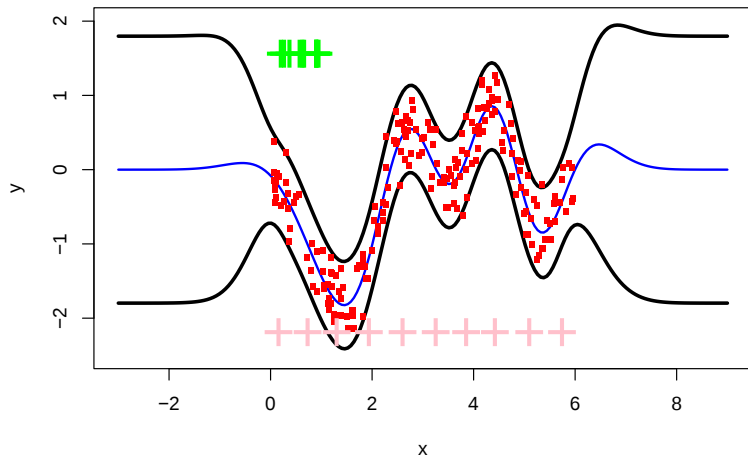
VFE: Illustrative Example

Full GP



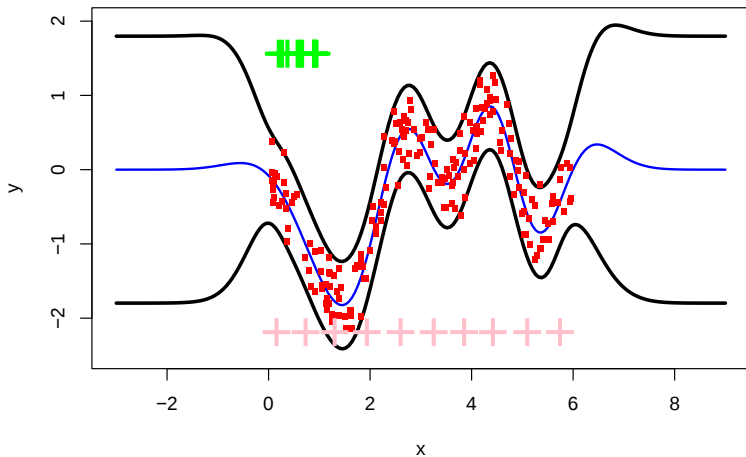
VFE: Illustrative Example

VFE ($M = 10$)



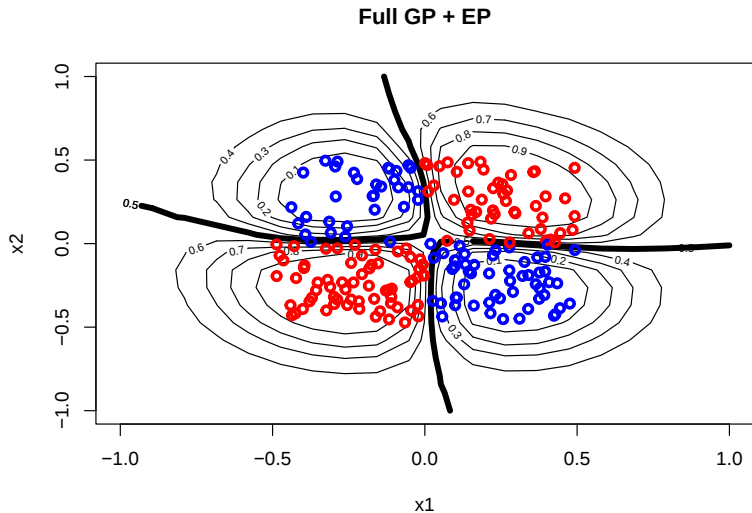
VFE: Illustrative Example

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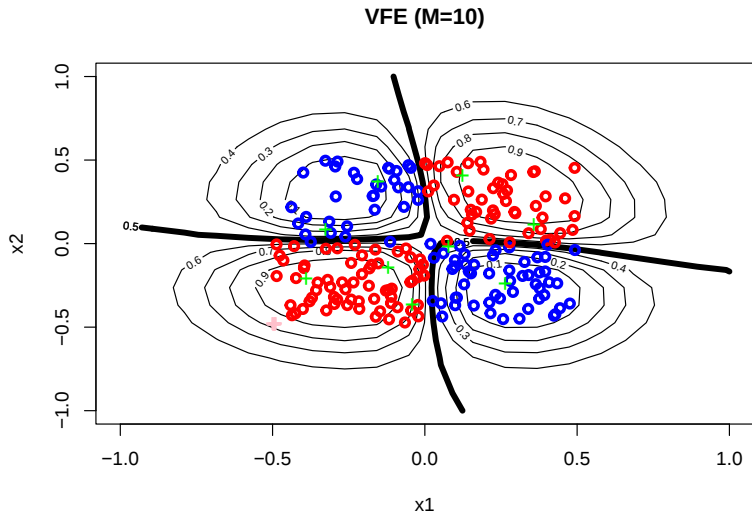


The inducing points cover the regions where the function changes!

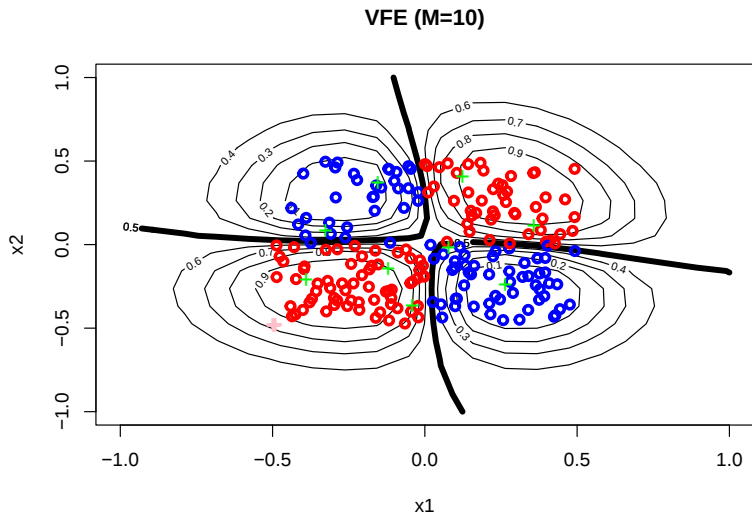
VFE: Illustrative Classification Example



VFE: Illustrative Classification Example



VFE: Illustrative Classification Example



The inducing points spread across the input space!

FITC vs. VFE

Two approaches:

FITC vs. VFE

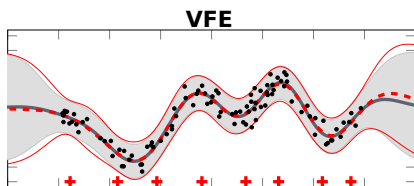
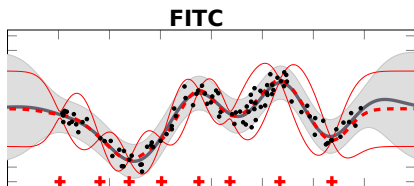
Two approaches:

- FITC: optimize the marginal likelihood of an approximate GP model.
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FITC vs. VFE

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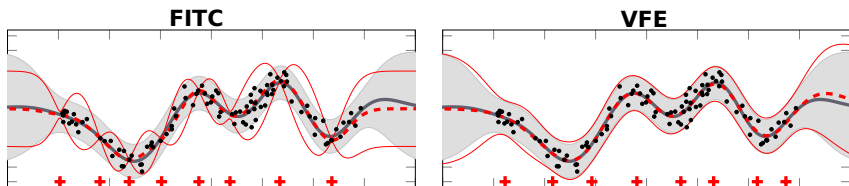
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FITC vs. VFE

Two approaches:

- FITC: optimize the marginal likelihood of an approximate GP model.
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- FITC: less local optima and easier to optimize, also less accurate.
- VFE: more accurate, more local optima, more difficult to optimize.

(Bui et al., 2017) (Bauer et al., 2016)

Run the cells of the notebook to fit a sparse GP using the VFE approximation and complete task 6!

Whitened Parameterization for VFE

Alternative VFE objective expected to be easier to optimize!

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Instead of making inference about $\mathbf{u}$, the whitened VFE objective makes inference about:

$$\mathbf{e} \quad \text{such that} \quad \mathbf{u} = \mathbf{L}\mathbf{e}, \quad \mathbf{e} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}),$$

with $\mathbf{u}$ the latent process values at the inducing points and $\mathbf{L}^T \mathbf{L} = \Sigma_{\mathbf{u}\mathbf{u}}$.

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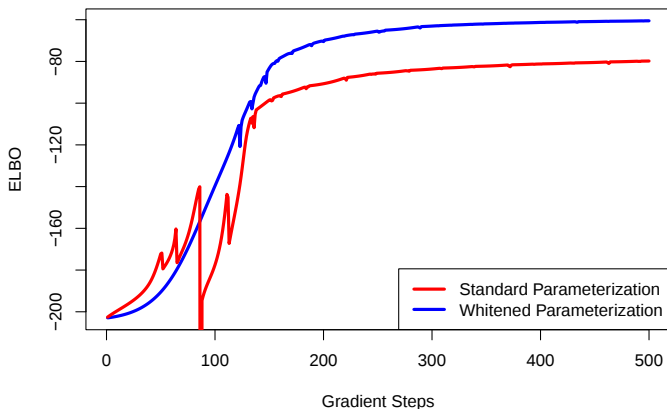
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The VFE objective becomes:

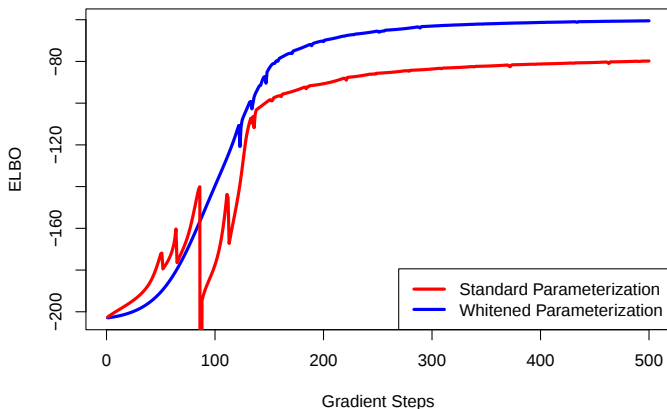
$$\sum_{i=1}^N \mathbb{E}_{q(\mathbf{e})p(f(\mathbf{x}_i)|\mathbf{e})} [\log p(y_i|f(\mathbf{x}_i))] - \text{KL}(q(\mathbf{e})|\mathcal{N}(\mathbf{0}, \mathbf{I})),$$

with $p(f(\mathbf{x}_i)|\mathbf{e})$ using the covariances between $f(\mathbf{x}_i)$ and $\mathbf{e}$.

Whitened Parameterization: Illustrative Example



Whitened Parameterization: Illustrative Example



Whitening significantly improves convergence!

**Run the cells of the notebook to fit a sparse GP
using the VFE approximation with whitening!**

Natural Gradient Ascent

Gradient ascent moves in the direction of the gradient $\nabla_{\xi} \mathcal{L}(\xi)$.

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Formally:

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If ξ represents the parameters of probability distributions, the Euclidean norm may be problematic!

Illustration with Two Gaussians

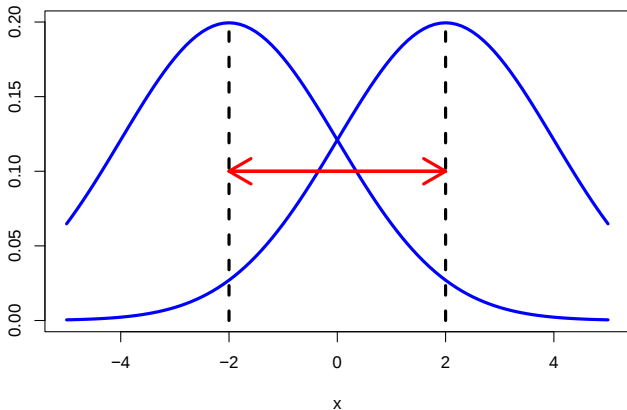


Illustration with Two Gaussians

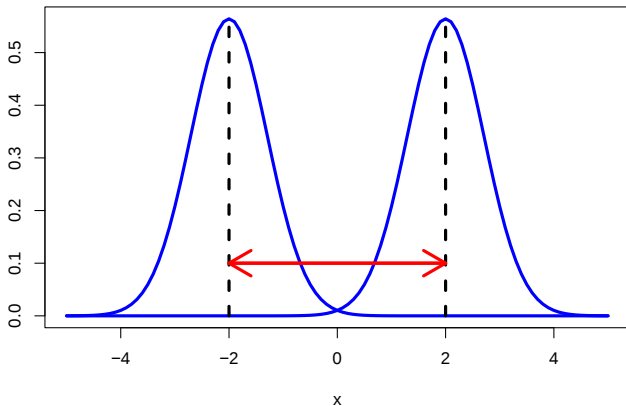
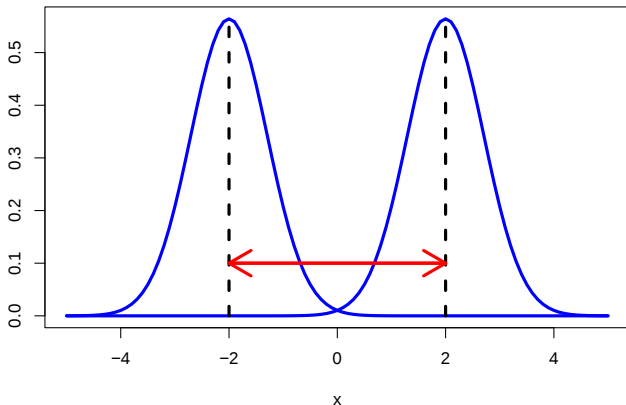
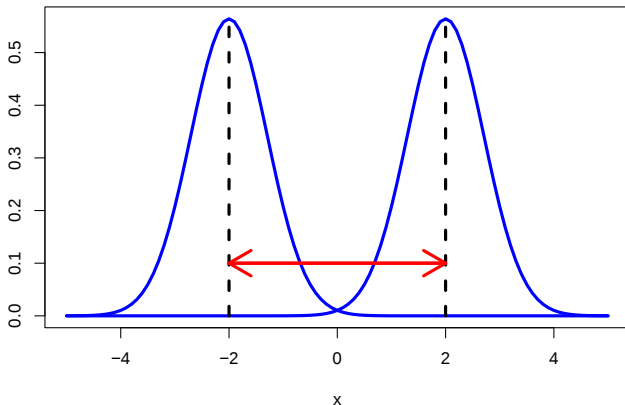


Illustration with Two Gaussians



The Euclidean distance between parameters is 4 in both cases!

Illustration with Two Gaussians



A better alternative is the KL-divergence between distributions!

Natural Gradient Ascent

Considers the KL-divergence as a norm:

$$\nabla_{\xi} \mathcal{L}(\xi) \mathbf{F}_{\xi}^{-1} \propto \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \arg \max_{\mathbf{d} \text{ s.t. } \text{KL}[q(\mathbf{u}|\xi)|q(\mathbf{u}|\xi+\mathbf{d})] \leq \epsilon} \mathcal{L}(\xi + \epsilon \mathbf{d})$$

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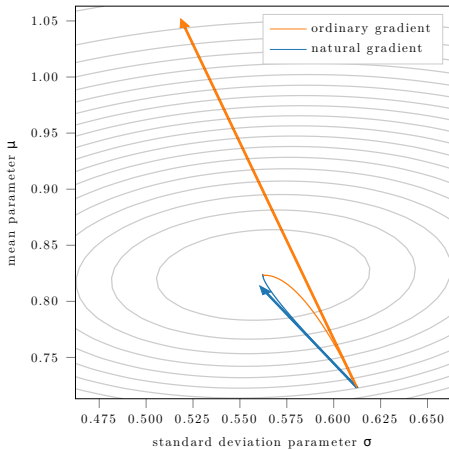
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Thus,

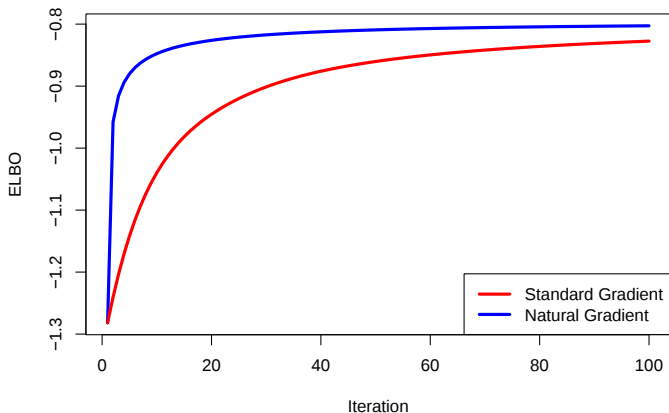
$$\nabla_{\xi} \mathcal{L}(\xi) \mathbf{F}_{\xi}^{-1} = \frac{\partial \mathcal{L}}{\partial \theta} \left(\frac{\partial \xi}{\partial \eta} \right)^{\top}.$$

Natural Gradient Ascent

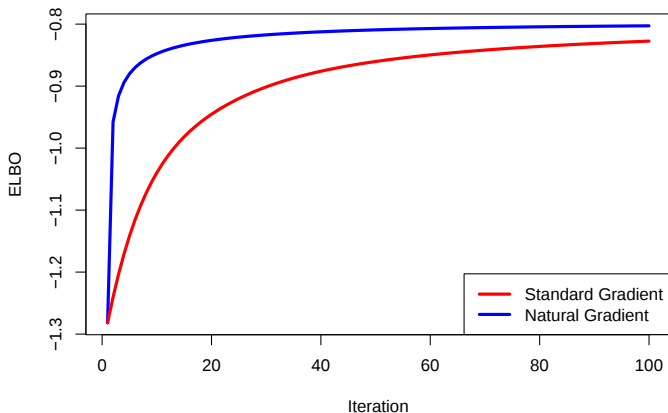


(Salimbeni et al., 2018)

Natural Gradient: Illustrative Example



Natural Gradient: Illustrative Example



The natural gradient achieves a faster convergence!

**Run the cells of the notebook to fit a sparse GP
using the VFE approximation with natural
gradients!**

GPs for Big Data

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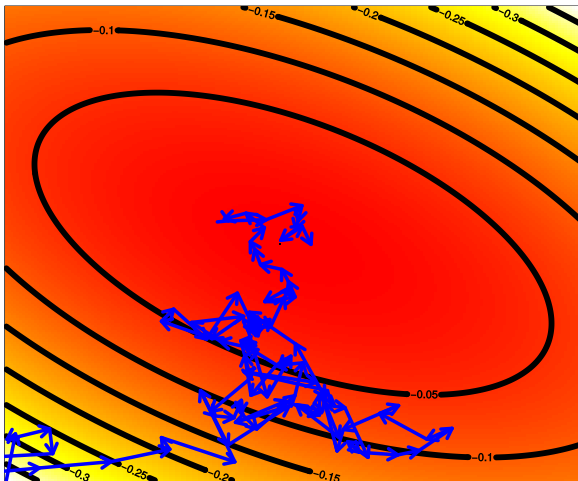
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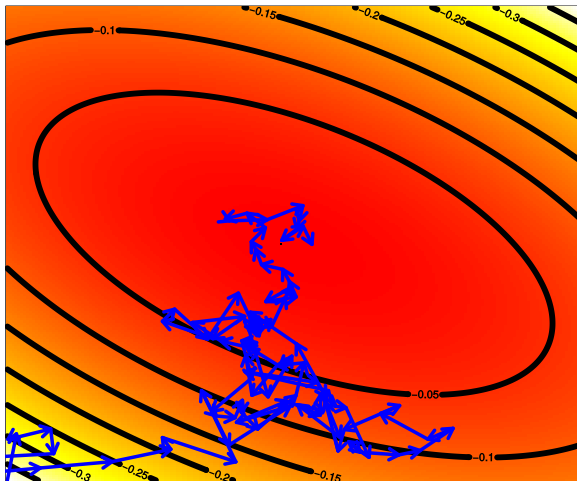
The training cost goes down to $\mathcal{O}(M^3)$ which allows to address datasets with millions of instances!

(Hensman et al., 2013)

GPs for Big Data

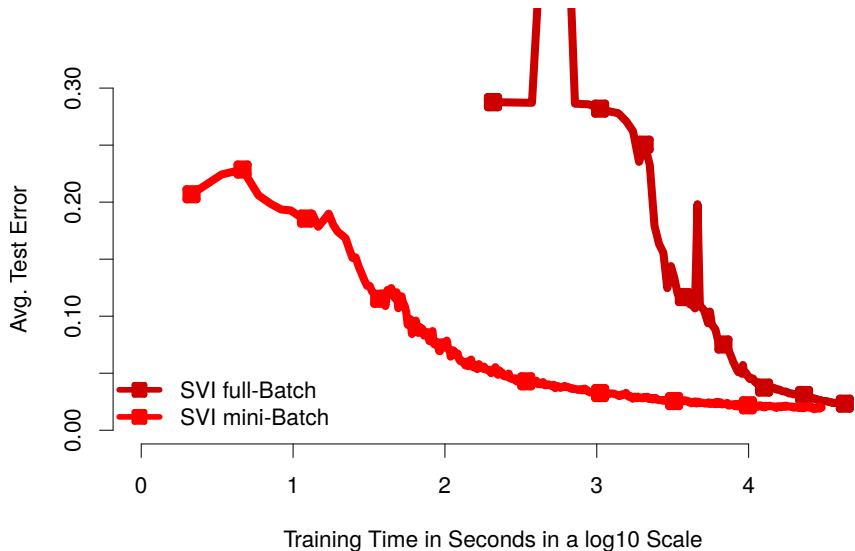


GPs for Big Data



To converge to a local neighborhood of the optimum stochastic methods require an estimate of the gradient which can be very cheap!

GPs for Big Data



(Hernández-Lobato, 2015)

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- Reduces the cost to $\mathcal{O}(MN^2)$ with $M \ll N$.

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- It allows for minibatch training which reduces the cost to $\mathcal{O}(M^3)$.
- The objective is prone to local optima and difficult to optimize.

Run the cells of the notebook to fit a sparse GP using the VFE approximation with mini-batches!

Run the cells of the notebook to fit a sparse GP using the VFE approximation for classification and carry out task 7!

Sparse GP Conclusions

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- The best performing method seems to be the VFE method since it does not modify the prior.
- Some methods allow for stochastic optimization and mini-batch training that further reduce the cost to $\mathcal{O}(M^3)$.

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