Department of Signal Theory and Communications

Image representation and processing with level sets

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Outline

- Image representation with level sets
  - Level set representation of images
  - Curve parametrization and evolution
- Image filtering
  - Curve evolution and image diffusion
  - Classical diffusion processes
- Image segmentation
  - Introduction to segmentation
  - Segmentation with active contours

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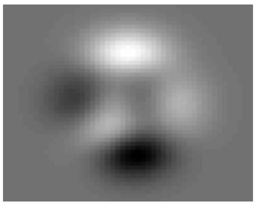
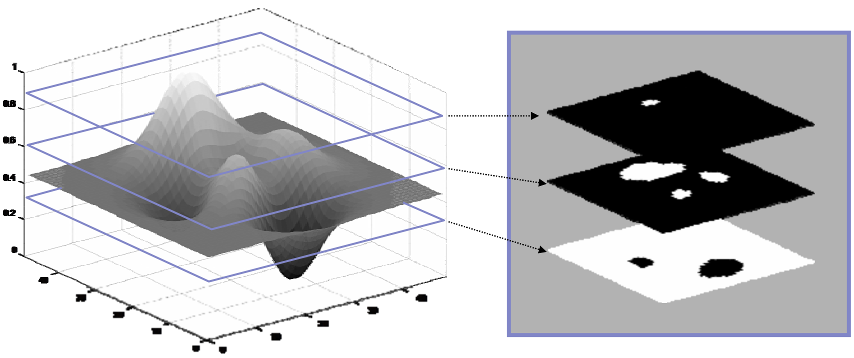
Level set representation of images (I)

- Consider the image as the stacking of geometrical primitives: its level sets





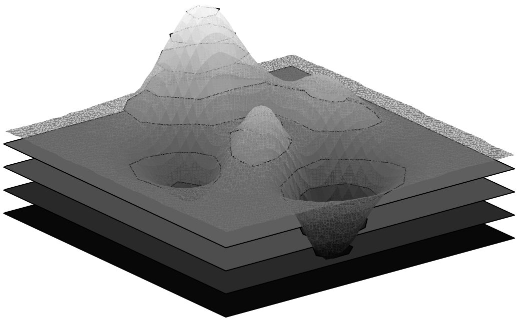
Level set representation of images (II)

- Decomposition of images in its level set:

$$X_\lambda = \{(i,j) \mid x(i,j) \geq \lambda\}$$

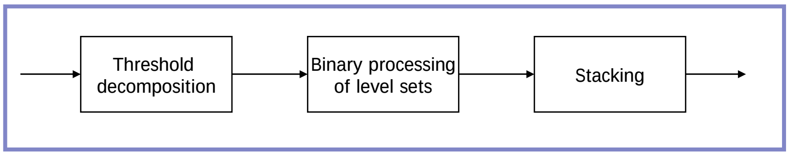
- Reconstruction of an image from its level sets (stacking):

$$x(i,j) = \vee\{\lambda \mid (i,j) \in X_\lambda\}$$



Level set representation of images (III)

- The representation highlights the geometrical aspects of:
  - The signal = the signal is the stacking of geometrical primitives.
  - The processing = **binary**, one can process the level sets and reconstruct the images.



- In the sequel: process images through the deformation of its level sets ⇔ Partial differential equations (PDE)

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Curve parametrization

- Continuous image model: i,j $\in$ R
- Curve parametrization:

$$C(p) = \{(i(p), j(p)), p \in [a,b]\}$$

- Arc–length parametrization:

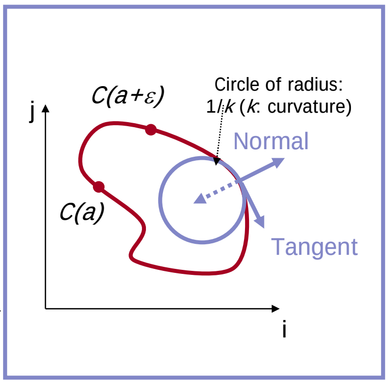
$$s \text{ such that } \left\| \frac{dC}{ds} \right\| = 1$$

$$C(s) = \{(i(s), j(s)), s \in [0, Length]\}$$

- Tangent and Normal:

$$\vec{T} = \frac{dC}{ds} = \left(\frac{di(s)}{ds}, \frac{dj(s)}{ds} \right)$$

$$k\vec{N} = \frac{d^2C}{ds^2}$$



$$\left\langle \frac{dC}{ds}, \frac{dC}{ds} \right\rangle = 1 \Rightarrow \left| \begin{array}{l} \left\langle \frac{dC}{ds}, \frac{d^2C}{ds^2} \right\rangle = 0 \\ \vec{T} \perp \frac{d^2C}{ds^2} = k\vec{N} \end{array} \right.$$

Level set and image features (I)

- Level set and level curve:

$$X_\lambda = \{(i,j) | x(i,j) \geq \lambda\},$$
$$C_\lambda = \{(i,j) | x(i,j) = \lambda\},$$

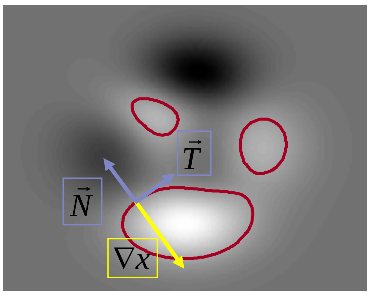
- The normal to the level curve and the image gradient are collinear:

$$\text{Along } C_\lambda,\ x(s) = \lambda \Rightarrow \frac{dx}{ds} = 0$$

$$\text{using the chain rule :}$$

$$\frac{dx}{ds} = \frac{\partial x}{\partial i}\frac{di}{ds} + \frac{\partial x}{\partial j}\frac{dj}{ds} = \left\langle \nabla x, \vec{T} \right\rangle = 0$$

$$\vec{N} = -\frac{\nabla x}{\|\nabla x\|}$$



Level set and image features (II)

- Curvature of level curve:
  - It can be shown that the curvature of level sets can also be computed directly on the image:

$$k = div\left(\frac{\nabla x}{\|\nabla x\|}\right) \qquad \text{"divergence"}: div(a) = \frac{da}{di} + \frac{da}{dj}$$

$$= \left(\frac{d^2x}{di^2}\frac{dx}{dj}^2 - 2\frac{dx}{di}\frac{dx}{dj}\frac{d^2x}{didj} + \frac{d^2x}{dj^2}\frac{dx}{di}^2\right) \Big/ \|\nabla x\|^3$$

$$\vec{N} = -\frac{\nabla x}{\|\nabla x\|}, \quad k = div\left(\frac{\nabla x}{\|\nabla x\|}\right), \quad \vec{T} = -\frac{\nabla x^{\perp}}{\|\nabla x\|}$$

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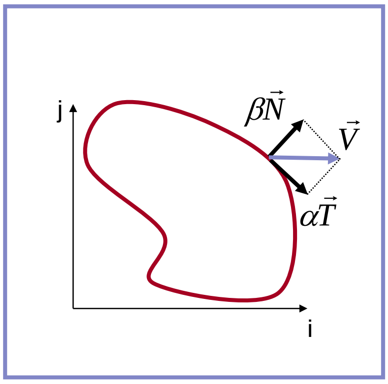
Curve evolution and image diffusion (I)

- Curve evolution through PDE:

$$\frac{\partial C}{\partial t} = \vec{V} = \alpha\vec{T} + \beta\vec{N}$$

- α simply modifies the curve parametrization. If we are interested in the evolution of the geometry, the most general evolution is given by:

$$\boxed{\frac{\partial C}{\partial t} = \beta\vec{N}}$$



Curve evolution and image diffusion (II)

- The fundamental level set equation:

$$\frac{\partial C}{\partial t} = \beta \vec{N} \quad \Leftrightarrow \quad \frac{\partial x}{\partial t} = \beta \left\| \nabla x \right\|$$

- Level set propagation with normal velocity β is equivalent to image diffusion by a PDE
  - Proof:

$$\text{Along}\, C_\lambda,\;\; x(i,j;t) = \lambda \Rightarrow \frac{dx(i,j;t)}{dt} = \frac{\partial x}{\partial i}\frac{di}{dt} + \frac{\partial x}{\partial j}\frac{dj}{dt} + \frac{\partial x}{\partial t} = 0$$

$$\frac{\partial x}{\partial t} = -\left\langle \nabla x, \frac{dC_\lambda}{dt} \right\rangle = -\left\langle \nabla x, \beta \vec{N} \right\rangle = -\beta \left\langle \nabla x, \vec{N} \right\rangle, \;\; \text{but}\; \vec{N} = -\frac{\nabla x}{\left\| \nabla x \right\|}$$

$$\frac{\partial x}{\partial t} = -\beta \left\langle \nabla x, -\frac{\nabla x}{\left\| \nabla x \right\|} \right\rangle = \beta \left\langle \nabla x, \frac{\nabla x}{\left\| \nabla x \right\|} \right\rangle = \beta \left\| \nabla x \right\|$$

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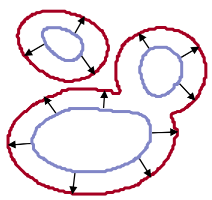
Classical diffusion processes

- Flows of constant velocity
- Smoothing flows:
  - Isotropic diffusion
  - Anisotropic diffusion
- Sharpening flow

Flows of constant velocity (I)

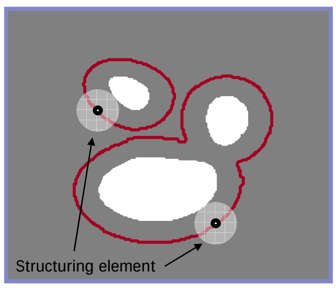
$$\boxed{\frac{\partial C}{\partial t} = \vec{N}}$$



- β = 1, flow in the direction of the level curve normal and of magnitude 1:

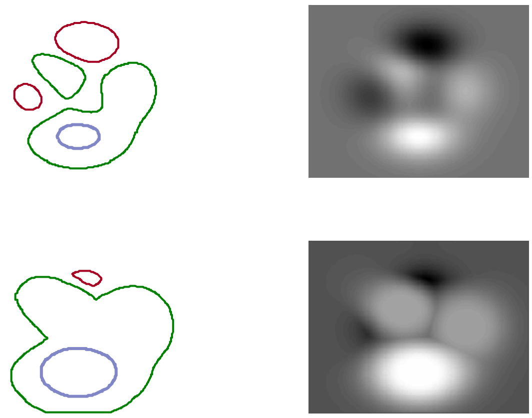
$$\frac{\partial C}{\partial t} = \vec{N} \quad \Leftrightarrow \quad \frac{\partial x}{\partial t} = \|\nabla x\|$$

- Equivalent to a dilation with a circle:
  - Isotropic expansion of constant velocity
  - Topological changes



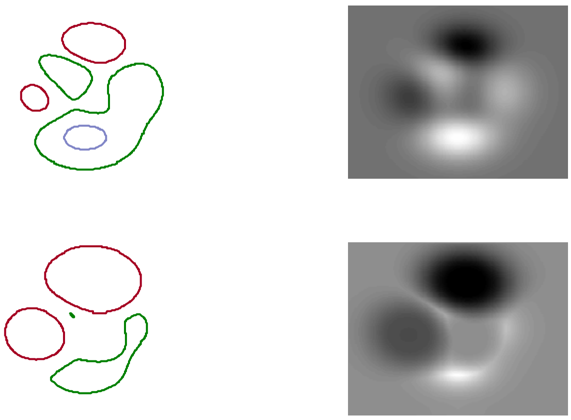
Flows of constant velocity (II)

$$\frac{\partial C}{\partial t} = \vec{N} \quad \Leftrightarrow \quad \frac{\partial x}{\partial t} = \|\nabla x\|$$



Flows of constant velocity (IV)

- Inverse flow: $$\frac{\partial C}{\partial t} = -\vec{N} \quad \Leftrightarrow \quad \frac{\partial x}{\partial t} = -\|\nabla x\|$$



Erosion

Isotropic smoothing flow (I)

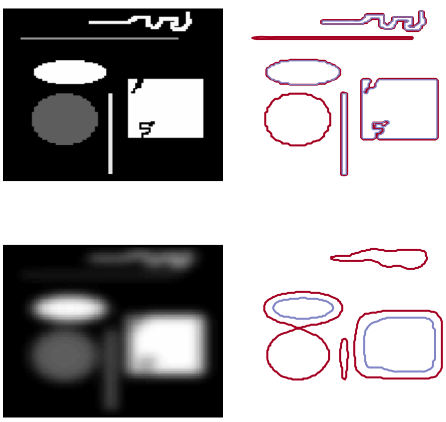
- In 1960 Gabor observed that the difference between a signal and its smoothed version is proportional to the laplacian of the signal.
- Heat equation: $\boxed{\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial i^2} + \frac{\partial^2 x}{\partial j^2} = div(\nabla x)}$
- The solution of the heat equation is known analytically: convolution with a Gaussian impulse response.

Isotropic smoothing flow (II)

$$\frac{\partial x}{\partial t} = div(\nabla x)$$

$$= \frac{div(\nabla x)}{\|\nabla x\|}\|\nabla x\|$$

- Level curves move according to: $\frac{div(\nabla x)}{\|\nabla x\|}$
- Smoothing is isotropic and goes across edges!



From isotropic to anisotropic flow (I)

- Goal: Smooth in a direction parallel to the edges

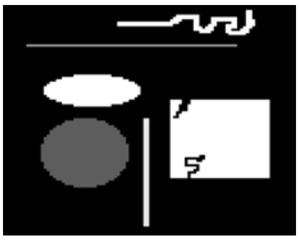
- Smoothing is stopped when curvature of level lines is low (edge presence):

$$\frac{\partial x}{\partial t} = div(\nabla x) = \frac{div(\nabla x)}{\|\nabla x\|}\|\nabla x\| \quad \text{changed to:}$$

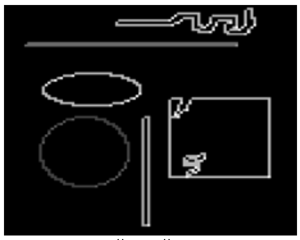
$$\frac{\partial x}{\partial t} = div\left(\frac{\nabla x}{\|\nabla x\|}\right)\|\nabla x\| = k\|\nabla x\|$$

$$\boxed{\frac{\partial x}{\partial t} = k\|\nabla x\| \quad \Leftrightarrow \quad \frac{\partial C}{\partial t} = k\vec{N}}$$

From isotropic to anisotropic flow (II)



x



$\|\nabla x\|$

$$div(\nabla x) = \frac{div(\nabla x)}{\|\nabla x\|}\|\nabla x\|$$

$$div\left(\frac{\nabla x}{\|\nabla x\|}\right)\|\nabla x\| = k\|\nabla x\|$$

From isotropic to anisotropic flow (III)

x

$\|\nabla x\|$

$$div(\nabla x) = \frac{div(\nabla x)}{\|\nabla x\|}\|\nabla x\|$$

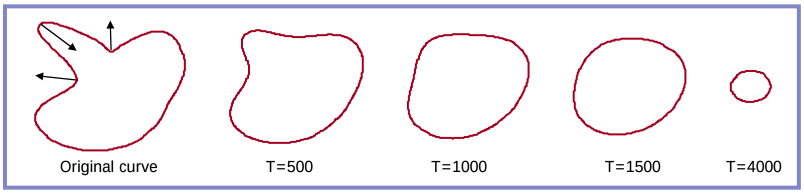
$$div\left(\frac{\nabla x}{\|\nabla x\|}\right)\|\nabla x\| = k\|\nabla x\|$$

Anisotropic diffusion: Curvature motion

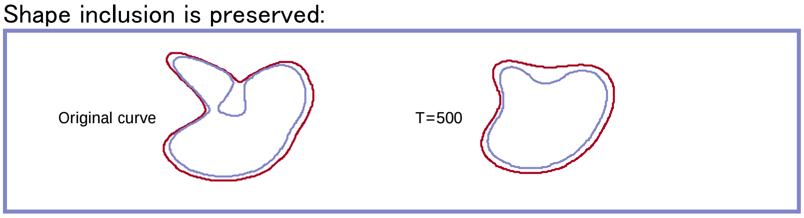
- Curvature motion is defined by the geometric heat flow:

$$\frac{\partial C}{\partial t} = k\vec{N} = \frac{d^2C}{ds^2}$$

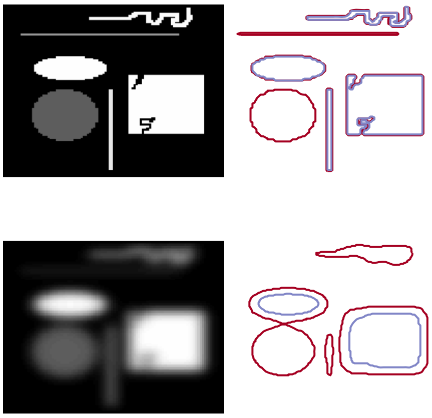
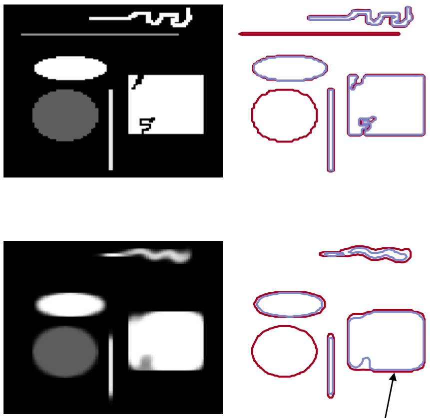
- Non convex shapes become convex (and then disappear):



- Shape inclusion is preserved:

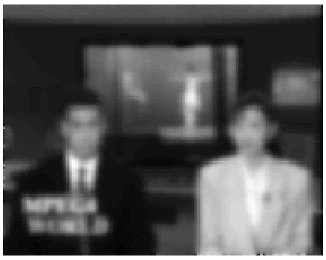


Anisotropic diffusion: image smoothing (I)

Isotropic diffusion	Anisotropic diffusion
	

Low curvature = reduced smoothing

Anisotropic diffusion: image smoothing (II)



Isotropic diffusion

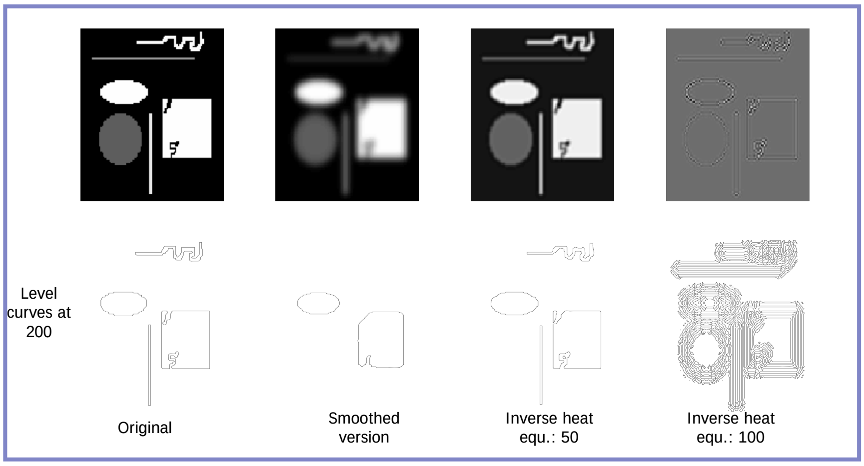
Anisotropic diffusion

Sharpening flow (I)

- Basic idea: The heat equation smoothes the signal. Can we invert it?

$$\frac{\partial x}{\partial t} = div(\nabla x) \quad \Rightarrow \quad \frac{\partial x}{\partial t} = -div(\nabla x)$$

Unstable process!



Sharpening flow (II)

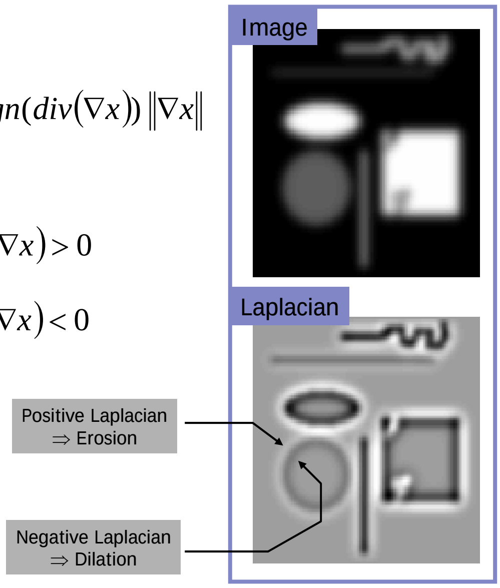
- Alternative strategy:

$$\frac{\partial x}{\partial t} = div(\nabla x) \Rightarrow \frac{\partial x}{\partial t} = -sign(div(\nabla x))\,\|\nabla x\|$$

That is:

$$\begin{cases} \dfrac{\partial x}{\partial t} = -\|\nabla x\|, & \text{when } div(\nabla x) > 0 \\ \dfrac{\partial x}{\partial t} = \|\nabla x\|, & \text{when } div(\nabla x) < 0 \end{cases}$$

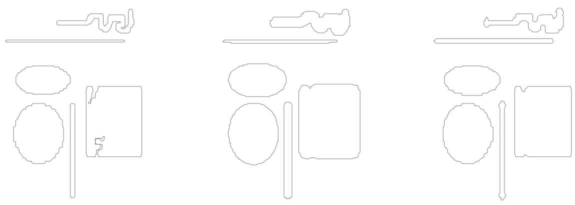
- Propagation stops when Laplacian = 0 (inflection points)
- Shock filters



Sharpening flow (III)

Original | Smoothed version | Shock filter

Level curves at 140



Sharpening flow (IV)

Original | Smoothed version | Shock filter

Summary on diffusion process

- Flow of constant velocity ⇔ Erosion, Dilation
- Isotropic diffusion ⇔ Heat equation (on the image)
- Anisotropic diffusion
  ⇔ Geometrical heat equation (on the level curve)
  ⇔ Curvature motion
- Shock filters
  ⇔ "Stable" inverse of the heat equation
  ⇔ Combination of erosion / dilation depending on the laplacian sign

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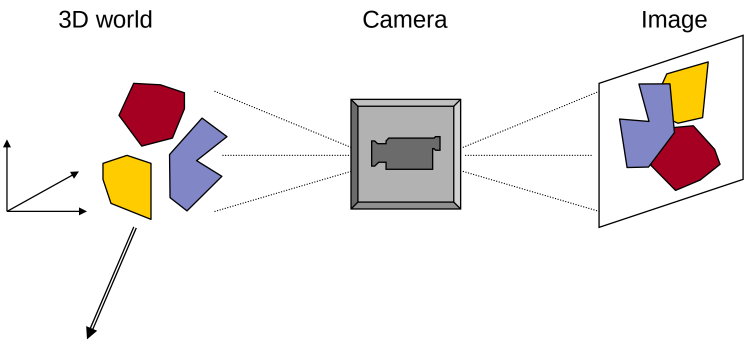
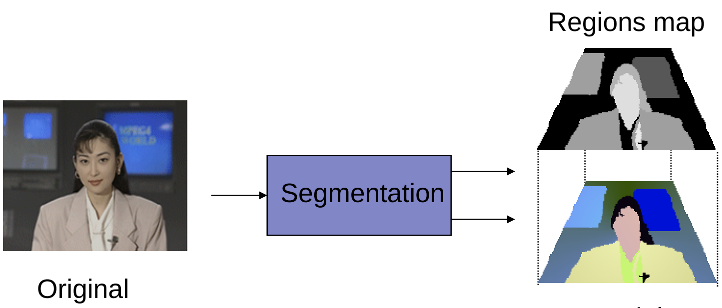
Objects and superposition model



Object: Shape
Texture (Gray level, color)

The contour-texture model



Hierarchical representation

Ideal definition of segmentation

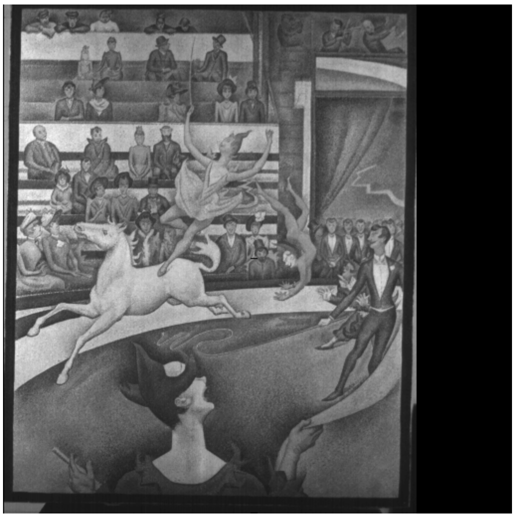
- Create a partition of image I: $\{R_i\}$

$$\begin{cases} \bigcup R_i = \text{Image} \\ R_i \cap R_j = \varnothing, \quad \forall i, j \end{cases}$$

... of connected regions R_i, R_j such that: (C: criterion)

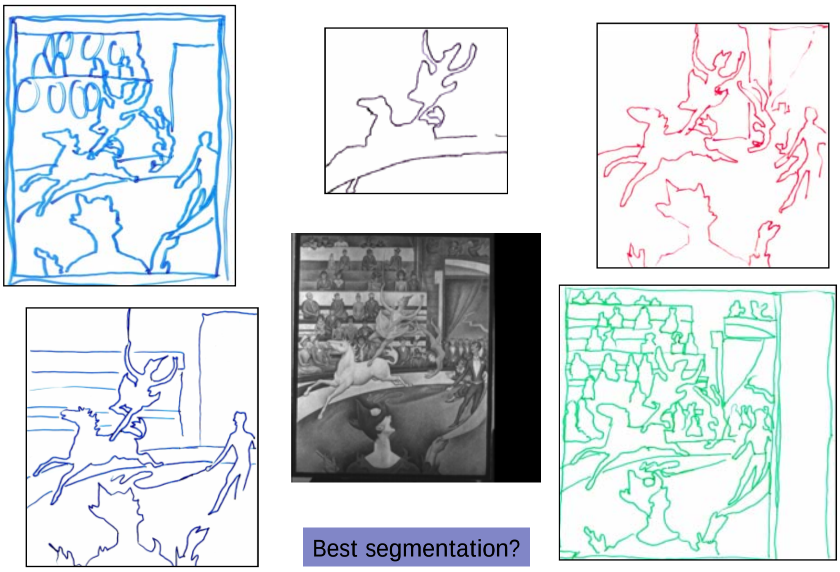
$$\begin{cases} C(R_i) = \text{True}, \quad \forall i \\ C(R_i \cup R_j) = \text{False}, \quad \forall i, j \end{cases}$$

Evaluation of the segmentation quality



- (Extremely) difficult
- Application dependant
- Often deceiving

Manual segmentation



General segmentation scheme

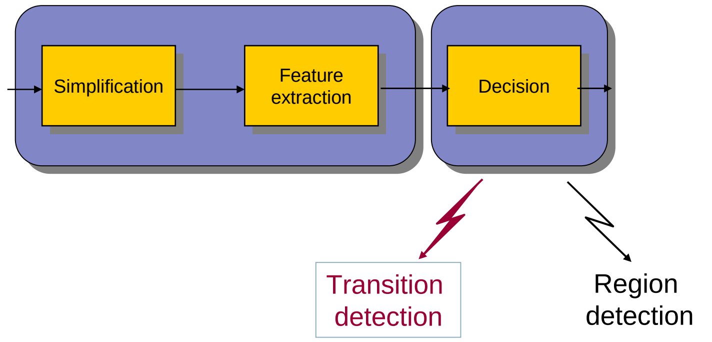
→ Simplification → Feature extraction → Decision →

Simplification
- Remove useless or annoying information
- Preserve shape information

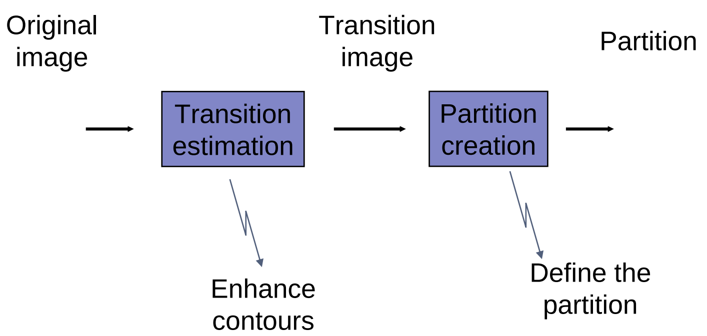
Feature extraction
- Feature space:
  - Gray level
  - Color
  - Variance
  - Frequency component
- Pixels belonging to the same regions should correspond to homogeneous features

Decision
- Partition definition:
  - → Transition
  - → Homogeneity

Segmentation: transition-based



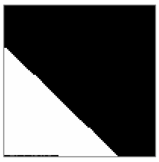
Transition detection



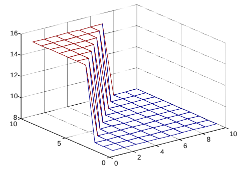
Transition estimation

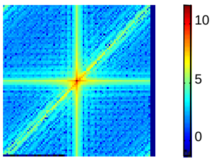
Which model ?

- Space-frequency → High frequencies → Simple differences, Prewitt, Sobel, etc.
- Geometric → Vertical transitions → Morphological gradients

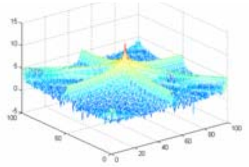


Original image



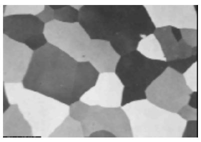


Spectrum

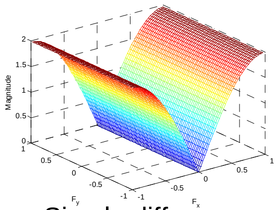


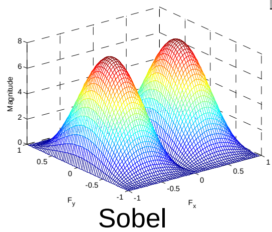
Examples of gradient estimation

Original image

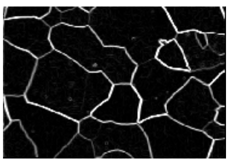


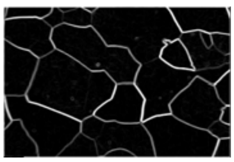
Frequency response

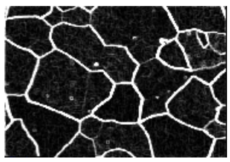




Simple differences | Sobel | Morphological





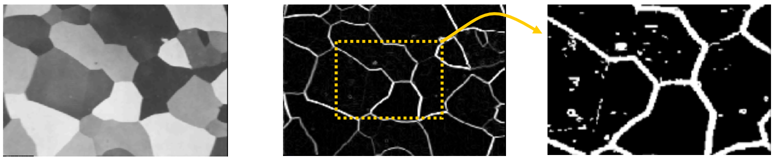


$||\nabla x||=Sqrt(\nabla_{hor}x^2+\nabla_{ver}x^2)$

$(x \oplus b) - (x \ominus b)$

Partition creation

- Straightforward strategy: Gradient binarization



  - Difficulty to define the threshold
  - Noise may appear: Cleaning with opening
  - Contours may not be closed: Closing or gradient extrapolation
  - Contours may be thick: remove pixels without breaking the contours
  - Not very robust!

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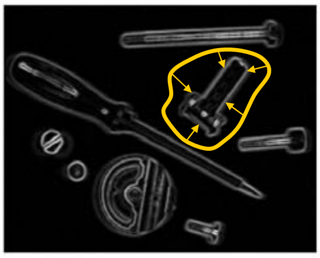
Partition creation

- Robust strategy: Active contours (Snakes)
  - Evolution of a close curve towards the points of high gradients



  - How to define the curve evolution?
  - How to implement it?
  - How to define the initial curve?

Active contour (snakes): curve evolution (I)

- Basic idea: Minimize the "length" of the curve
  - If s is the arc-length parametrization of the curve C:

$$\text{Length} = \oint ds$$

$$\text{Geodesic length} = \oint g(\nabla x)ds, \text{where g is function of } \nabla x$$

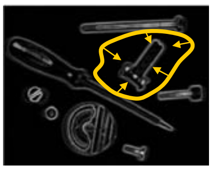
  - Assume g is low when the gradient is high and $g \in [0,1]$, ex:

$$g(\nabla x) = \frac{1}{1+\|\nabla x\|^p}$$

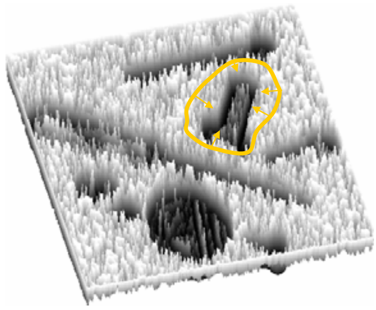
Active contour: curve evolution (II)

Find the curve $C(s)$ such that $\oint g(\nabla x)ds$ is minimum

Gradient: ∇x



3D view of ∇x



3D view of g(∇x)

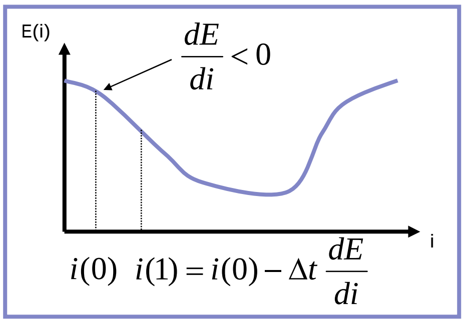
Active contour: curve evolution (III)

- How to find the minimum of a function?:
  - Compute the gradient of the function:

$$\text{Minimum of } E(i) \quad \Rightarrow \quad \frac{dE(i)}{di} = 0$$

  - Gradient descent strategy: iterative search of i ⇒ i(t). It leads to the following PDE:

$$\frac{\partial i(t)}{\partial t} = -\frac{\partial E(i)}{\partial i}$$



Active contour: curve evolution (IV)

- Using as criterion the geodesic length:

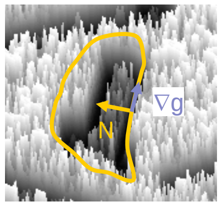
$$E(C)=\oint g(\nabla x)ds \;\Rightarrow\; \frac{\partial C}{\partial t}=-\frac{\partial E}{\partial C}=gk\vec{N}-\left\langle \nabla g,\vec{N}\right\rangle \vec{N}$$

Internal force:
- Curvature motion
- Smoothing

External force:
- The evolution stops when the gradient of g is perpendicular to the curve normal

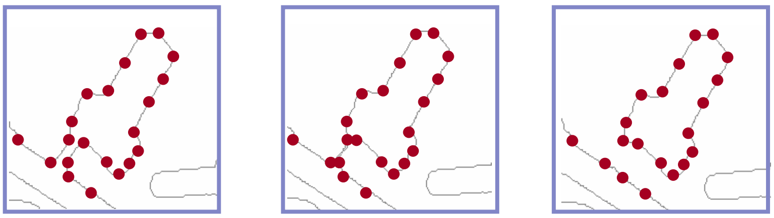
Add erosion/dilation to search the space:

$$\frac{\partial C}{\partial t}=g(k+c)\vec{N}-\left\langle \nabla g,\vec{N}\right\rangle \vec{N}$$



Active contour: implementation (I)

- Explicit implementation is complex:
  - Resample the curve when it shrinks
  - Handle complex topological changes



- Implicit implementation is much simpler:
  - Create a (any) function *u* which has *C(s)* as zero level set.
  - Use the level set result:

$$\frac{\partial C}{\partial t}=\beta\vec{N} \;\Leftrightarrow\; \frac{\partial u}{\partial t}=\beta\left\|\nabla u\right\|$$

No curve sampling, no curve resampling, natural handling of topological changes

Active contour: implementation (II)

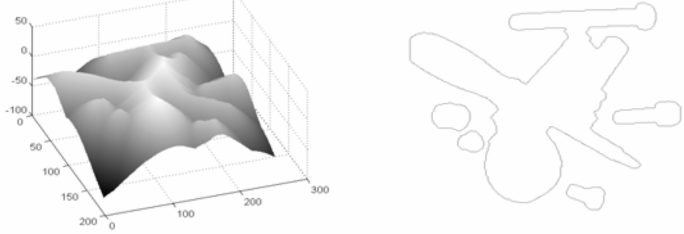
$$\frac{\partial C}{\partial t} = g(k+c)\vec{N} - \left\langle \nabla g, \vec{N} \right\rangle \vec{N}$$

$$\Downarrow$$

$$\frac{\partial u}{\partial t} = g(k+c)\|\nabla u\| + \left\langle \nabla g, \frac{\nabla u}{\|\nabla u\|} \right\rangle \|\nabla u\|, \quad \begin{cases} \vec{N} = -\nabla u / \|\nabla u\|, \text{ and} \\ k = div\left(\nabla u / \|\nabla u\|\right) \end{cases}$$

- Classical approach: *u* is the distance function of *C(s)*

Evolution of distance function ⇔ Evolution of the curve

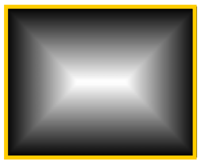


Active contour: example

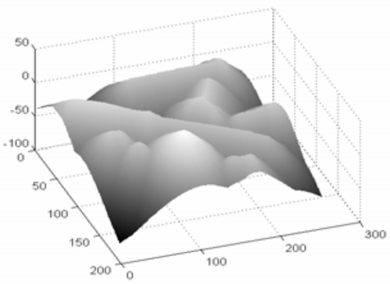
Original image *x*

Gradient ∇x

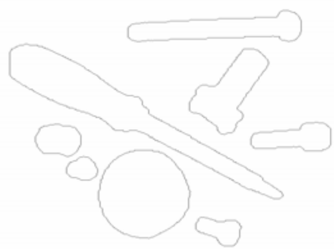
$g(\nabla x)$



Initial contour and distance function *u*



Final distance function *u*



Final contour

Summary (I)

- Level set representation of images:
  - Fundamental geometric representation leading to
    - Mathematical morphology (historical evolution)
    - Diffusion process on images:
      Evolution of level curve ⇔ PDE proportional to image gradient

- Classical diffusion processes:
  - Flow of constant velocity ⇔ Erosion, Dilation
  - Isotropic diffusion ⇔ Heat equation (on the image)
  - Anisotropic diffusion
    ⇔ Geometrical heat equation (on the level curve)
    ⇔ Curvature motion
  - Shock filters
    ⇔ "Stable" inverse of the heat equation
    ⇔ Combination of erosion / dilation depending on the laplacian sign

Summary: Transition-based segmentation

- 3 main steps for segmentation:
  - Simplification, feature extraction and decision
  - Decision can be based on discontinuity (transition-based) or homogeneity (region-based)

- Transition-based segmentation:
  - Estimate transitions: gradient (may be linear or not)
  - Decision:
    - Thresholding: difficult because of threshold definition, noise and open contour
    - Active contours:
      - Curve evolution to minimize the length "weighted" by the gradient
      - Gradient descent leads to a combination of internal (smoothing) forces and external force (attraction to data)
      - Use an implicit representation of the curve: level set implementation